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AN ALGEBRAIC CLASSIFICATION OF SOME EVEN-DIMENSIONAL SPHERICAL KNOTS. II

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ABSTRACT. The paper reduces the problem of classification of simple even-dimensional spherical knots of codimension two to an algebraic problem.

This paper is the continuation of [**1**]. Here is given a classification of simple even-dimensional spherical knots in terms of algebraic invariants of the complement's infinite cyclic covering. The concluding result is contained in Theorem 11.6.

Here I briefly outline the structure of the invariants obtained. Let (S^{2q+2}, k^{2q}) be a $(q-1)$-simple spherical knot and $p: \tilde{X} \to X$ be the infinite cyclic covering of its complement. Let A denote $H_q(\tilde{X})$ and B denote the stable homotopy group of $\tilde{X}$ in dimension $q+2$. The generator $t: \tilde{X} \to \tilde{X}$ of the covering transformation group of p induces some automorphisms of A and B and thus A and B get $\Lambda = \mathbf{Z}[t, t^{-1}]$-module structures. In §11 of this paper more invariants are defined: forms

$$l: T(A) \otimes_{\mathbf{Z}} T(A) \to \mathbf{Q}/\mathbf{Z}, \qquad \psi: B \otimes_{\mathbf{Z}} B \to \mathbf{Z}_4$$

and the Λ-homomorphism

$$\alpha: A \otimes_{\mathbf{Z}} \mathbf{Z}_2 \to B$$

where $T(A)$ denotes the $\mathbf{Z}$-torsion subgroup of A. The main Theorem 11.6 states that the collection of invariants (A, B, α, l, ψ) completely determines the type of a simple even-dimensional spherical knot. In this theorem there are also described all properties to which the sets (A, B, α, l, ψ) coming from simple even-dimensional knots satisfy.

The whole work consists of 11 sections:

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The history of the subject, the brief characteristic of the previous works and §§1–4 are given in [**1**]. The concluding result of [**1**] is Theorem 4.2 which reduces the classification problem of simple even-dimensional spherical knots to a purely algebraic one: to classify P-quintets up to R-equivalence. The last problem is solved in §§5–11 presented in this paper and its solution is a transfer from P-quintets to Λ-quintets and, accordingly, the R-equivalence relation is substituted by the isomorphism. The general scheme of subsequent bijections constructed in [**1**] and in the present paper is as follows:

$$\left\{\begin{array}{l}\text{the types of } (q-1)\text{-simple}\\ 2q\text{-dimensional spherical}\\ \text{knots}\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\text{the } R\text{-equivalence classes}\\ \text{of } 2q\text{-isometries on virtual}\\ \text{complexes of length} \leqslant 2\end{array}\right\}$$
$$\leftrightarrow \left\{\begin{array}{l}\text{the } R\text{-equivalence classes}\\ \text{of } P\text{-quintets of parity}\\ (-1)^{q+1}\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\text{the isomorphism classes of}\\ L\text{-quintets of parity}\\ (-1)^{q+1}\end{array}\right\}$$
$$\leftrightarrow \left\{\begin{array}{l}\text{the isomorphism classes}\\ \text{of } \Lambda\text{-quintets of parity}\\ (-1)^{q+1}\end{array}\right\}.$$

Here q is any fixed integer, $q \geqslant 4$.

The aims of this paper are similar to those of Trotter's work [**15**] where the bijection between S-equivalence classes of Seifert matrices and congruence classes of Blanchfield forms was constructed. This work of Trotter greatly influenced mine but unlike [**15**] I do not use p-adic completions. Moreover, I have found a brief proof of Trotter's theorem [**15**] which also does not use p-adic completions. This proof I plan to publish separately.

In this paper only those results are included which take part in the proof of Theorem 11.6. However, many of them, to my mind, are of independent interest. To such results I would ascribe the following: finding of the connection between stable knots and isometries (§2); the study of knot modules (§§5, 6); the development of the techniques of isometry modifications (§8); and also the establishing of the formula which gives the holomogy of the infinite cyclic covering through the homology of a Seifert manifold (§10).

The terminology and notation of this paper are as in [**1**]. References to [**1**] are not indicated here; for example, when it is said about Theorem 4.2, it means the Theorem 4.2 from [**1**] and so on. As in [**1**] the formulas are numerated anew in each section.

5. Knot modules. In this section we establish a connection between knot modules and finitely-generated over $\mathbf{Z}$ modules over the integer polynomial ring. The obtained result is similar to those of Trotter [**15**], where it was assumed that knot modules were supplied with the Blanchfield pairing which only occured on the knot module of the middle dimension. In §7 the result of this section will be applied to obtain a theorem of the same kind for the more difficult case of P-quintets.

The geometrical interpretation of the results of this section will be given below in §10.

5.1. Let $\bar{z} = 1 - z$, $P = \mathbf{Z}[z]$. Two finitely-generated over $\mathbf{Z}$, P-modules A and B will be called *contiguous* if there exist P-homomorphisms $\varphi\colon A \to B$ and $\hat{\varphi}\colon B \to A$ such that both $\varphi \circ \hat{\varphi}$ and $\hat{\varphi} \circ \varphi$ coincide with the multiplications by $z(1 - z) = z \cdot \bar{z} \in P$. The contiguity is a symmetric and reflexive relation. The equivalence relation generated by the contiguity will be called *R-equivalence*. More precisely, two finitely-generated over $\mathbf{Z}$, P-modules A and B are called R-equivalent if there is a finite sequence $A_1, A_2, \ldots, A_m$ of finitely-generated over $\mathbf{Z}$, P-modules with $A_1 = A$, $A_m = B$, and A_i being contiguous to A_{i+1} for $i = 1, \ldots, m - 1$.

To state the main result of this section let us introduce the ring $L = \mathbf{Z}[z, z^{-1}, \bar{z}^{-1}]$. The ring P is a subring in L and so L is a P-module.

5.2. THEOREM. *Any two finitely-generated over* $\mathbf{Z}$, *P-modules A and B are R-equivalent if and only if the L-modules $A \otimes_P L$ and $B \otimes_P L$ are isomorphic.*

Let $\Lambda = \mathbf{Z}[t, t^{-1}]$ be the ring of integer Laurent polynomials. We shall consider Λ as a subring in L, thinking that t is equal to $1 - z^{-1}$. Then $L = \Lambda[(1 - t)^{-1}]$ and so any Λ-module, for which multiplication by $1 - t$ is an isomorphism, has the natural L-module structure. Conversely, any L-module defines a Λ-module with multiplication by $1 - t$ being isomorphic.

5.3. THEOREM. *The following conditions are equivalent*:

(a) *An L-module $\tilde{A}$ is isomorphic to $A \otimes_P L$ for some finitely-generated over* $\mathbf{Z}$, *P-module A*;

(b) *an L-module $\tilde{A}$ is finitely-generated over* $\mathbf{Z}[z^{-1}, \bar{z}^{-1}]$;

(c) *$\tilde{A}$ is a finitely-generated Λ-module and multiplication by* $(1 - t) \in \Lambda$ *is an isomorphism $\tilde{A} \to \tilde{A}$*;

(d) *$\tilde{A}$ is a finitely-generated Λ-module and there is an integer polynomial $\Delta(t)$ such that* $\Delta(1) = 1$ *and* $\Delta\tilde{A} = 0$.

Λ-modules satisfying (c), and hence other conditions of Theorem 5.3, are called *modules of type K* [**12**].

From Theorems 5.2 and 5.3 follows

5.4. THEOREM. *There exists one-to-one correspondence between the R-equivalence classes of finitely-generated over* $\mathbf{Z}$, *P-modules and classes of isomorphic modules of type K. This correspondence is given by* $A \mapsto A \otimes_P L$.

The rest of this section is devoted to the proofs of Theorems 5.2 and 5.3.

A P-module A will be called *minimal* if the multiplication by $z \cdot \bar{z} \in P$ is a monomorphism $A \to A$.

5.5. LEMMA. *The map $A \to A \otimes_P L$ which sends $a \in A$ to $a \otimes 1$ is a monomorphism if and only if A is a minimal P-module.*

The proof follows from the fact that L as P-module is isomorphic to the direct limit $P \to P \to P \to \cdots$, where all homomorphisms are multiplications by $z \cdot \bar{z}$.

5.6. LEMMA. *Any finitely-generated over* $\mathbf{Z}$, *P-module* A *is R-equivalent to some finitely-generated over* $\mathbf{Z}$, *minimal P-module.*

PROOF. Let $K \subset A$ be a submodule such that $z \cdot \bar{z} \cdot K = 0$. Let $A_1 = A/K$ and φ: $A \to A_1$ be the projection. Define $\hat{\varphi}$: $A_1 \to A$ by the formula $\hat{\varphi}(x) = z \cdot \bar{z} \cdot \varphi^{-1}(x)$, $x \in A_1$. This correctly defines $\hat{\varphi}$, and clearly, $\varphi \circ \hat{\varphi}$ and $\hat{\varphi} \circ \varphi$ coincides with multiplications by $z \cdot \bar{z}$.

So, if there is a nonzero submodule $K \subset A$ with $z \cdot \bar{z} \cdot K = 0$, then one can obtain a contiguous module A_1. If A_1 also contains some such submodule $K_1 \subset A_1$ then by the same way we obtain a module A_2 and so on. We shall have the sequence $A, A_1, A_2, \ldots$, where each module A_i is less than A_{i-1} in the sense that either the rank of A_i is less than the rank of A_{i-1} or the ranks are equal and the order of the torsion subgroup of A_i is less than the order of the torsion subgroup of A_{i-1}. From this it is clear that this sequence cannot be infinite. If A_j is its last module then A_j has no nonzero submodules $K \subset A_j$ with $z \cdot \bar{z} \cdot K = 0$. If $a \in A_j$ were an element for which $z \cdot \bar{z} \cdot a = 0$, $a \neq 0$, then the module K generated by a and za would be a submodule with $z \cdot \bar{z} \cdot K = 0$. So the module A_j is minimal and the lemma is proved.

5.7. PROPOSITION. *Any two finitely-generated over* $\mathbf{Z}$, *P-modules* A_1 *and* A_2 *are R-equivalent if and only if there is an integer* $n \geqslant 0$ *and P-homomorphisms* φ: $A_1 \to A_2$ *and* $\hat{\varphi}$: $A_2 \to A_1$ *such that both* $\varphi \circ \hat{\varphi}$ *and* $\hat{\varphi} \circ \varphi$ *are multiplications by* $(z\bar{z})^n$.

PROOF. Evidently, the R-equivalence implies the condition formulated in Proposition 5.7. So we shall prove only the inverse assertion. By virtue of Lemma 5.6 we may suppose that A_1 and A_2 are minimal.

Let $C = \operatorname{coker}(\varphi)$ and λ: $A_2 \to C$ be the canonical projection. We have $(z \cdot \bar{z})^n C = 0$. Let $K \subset C$ be the kernel of the homomorphism $C \to C$, which is the multiplication by $z\bar{z}$. Let us denote $B_1 = \lambda^{-1}(K)$. Let φ_1: $B_1 \to A_2$ be the inclusion and μ: $A_1 \to B_1$ be the map given by $\mu(a) = \varphi(a)$ for $a \in A_1$. We obtain the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & 0 \arrow[d] & 0 \arrow[d] & K \arrow[d, "\cap"] & \\
0 \arrow[r] & A_1 \arrow[r, "\varphi"] \arrow[d, "\mu"'] & A_2 \arrow[r, "\lambda"] \arrow[d, "i"] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B_1 \arrow[r, "\varphi_1"'] \arrow[d] & A_2 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\
 & X \arrow[d] & 0 & 0 & \\
 & 0 & & &
\end{tikzcd}$$

It is easy to see that X is isomorphic to K. Thus it follows that $z\bar{z}X = 0$. Therefore the homomorphism $\hat{\mu}$: $B_1 \to A_1$ is well defined by the formula $\hat{\mu}(x) = \mu^{-1}(z\bar{z}x)$. So A_1 and B_1 are contiguous.

Define also $\hat{\varphi}_1$: $A_2 \to B_1$ by $\hat{\varphi}_1(a) = (z\bar{z})^{n-1}a$, $a \in A_2$. This homomorphism takes values in B_1, since $(z\bar{z})^{n-1}C_1 = 0$. We have now that φ_1 and $\hat{\varphi}_1$ are monomorphisms and both $\varphi_1 \circ \hat{\varphi}_1$ and $\hat{\varphi}_1 \circ \varphi_1$ coincide with the multiplication by $(z\bar{z})^{n-1}$.

One can now apply the above described process to modules B_1 and A_2 and to homomorphisms φ_1 and $\hat{\varphi}_1$ and so on. We shall obtain the modules $B_1, B_2, \ldots, B_{n-1}$ and homomorphisms φ_i: $B_i \to A_2$, $\hat{\varphi}_i$: $A_2 \to B_i$ such that B_i is contiguous to B_{i+1} and both $\varphi_i \circ \hat{\varphi}_i$, $\hat{\varphi}_i \circ \varphi_i$ coincide with multiplication by $(z\bar{z})^{n-i}$. Then B_{n-1} is contiguous to A_2, and so A_1 and A_2 are R-equivalent.

5.8. PROOF OF THEOREM 5.2. Suppose the P-modules A and B are R-equivalent and finitely generated over Z. By Proposition 5.7 there are P-homomorphisms φ: $A \to B$ and $\hat{\varphi}$: $B \to A$ such that $\varphi \circ \hat{\varphi}$ and $\hat{\varphi} \circ \varphi$ coincide with multiplications by $(z\bar{z})^n \in P$ for some $n \geqslant 0$. Let Φ: $A \otimes_P L \to B \otimes_P L$ be $\varphi \otimes z^{-n}$ and $\hat{\Phi}$: $B \otimes_P L \to A \otimes_P L$ be $\hat{\varphi} \otimes \bar{z}^{-n}$. Then the compositions $\Phi \circ \hat{\Phi}$ and $\hat{\Phi} \circ \Phi$ are identities and thus Φ and $\hat{\Phi}$ are isomorphisms.

Inversely, suppose modules A and B are such that L-modules $A \otimes_P L$ and $B \otimes_P L$ are isomorphic. By Lemma 5.6 we may suppose that A and B are minimal. Then the natural maps

$$\alpha\colon A \to A \otimes_P L, \qquad \beta\colon B \to B \otimes_P L,$$

which act by the formula $x \mapsto x \otimes 1$, are monomorphisms (see Lemma 5.5). Let us identify $A \otimes_P L$ with $B \otimes_P L$ by means of some isomorphism and denote the result by C. $\operatorname{im}\alpha$ and $\operatorname{im}\beta$ are P-submodules in C. Let us show that for each $c \in C$ there is some $n \geqslant 0$ such that $(z\bar{z})^n c \in \operatorname{im}\alpha$. Really, for each finite system $l_1, l_2, \ldots, l_r \in L$ there is an $n \geqslant 0$ such that $l_i = (z\bar{z})^{-n}p_i(z)$, where $p_i(z)$ is some integer polynomial, $i = 1, 2, \ldots, r$. From this it is clear that if

$$c = \sum_{i=1}^{r} a_i \otimes l_i$$

where $a_i \in A$, then $(z\bar{z})^n c \in \operatorname{im}\alpha$. Since $\operatorname{im}\beta$ is finitely-generated, one can choose n such that $(z\bar{z})^n \operatorname{im}\beta \subset \operatorname{im}\alpha$. Similarly, there is an $m \geqslant 0$ such that $(z\bar{z})^m \operatorname{im}\alpha \subset \operatorname{im}\beta$.

Let us now define φ: $A \to B$ and $\hat{\varphi}$: $B \to A$ by the formulas

$$\varphi(a) = \beta^{-1}\big((\bar{z})^m \alpha(a)\big), \qquad \hat{\varphi}(b) = \alpha^{-1}\big((z\bar{z})^n \beta(b)\big),$$

where $a \in A$, $b \in B$. Then $\hat{\varphi}(\varphi(a)) = (z\bar{z})^{n+m}a$, $\varphi(\hat{\varphi}(b)) = (z\bar{z})^{n+m}b$. By Proposition 5.7 these imply that A and B are R-equivalent.

The theorem is proved.

5.9. PROOF OF THEOREM 5.3. Let $\tilde{A} = A \otimes_p L$, where A is finitely-generated over $\mathbf{Z}$. As it was noticed in the proof of Theorem 5.2, any element $a \in \tilde{A}$ can be presented in the form of $a = (z\bar{z})^{-n}a_1$, where $n \geqslant 0$, and a_1 belongs to the image of the homomorphism α: $A \to A \otimes_P L$, $\alpha(x) = x \otimes 1$, $x \in A$. The image of α is finitely generated over $\mathbf{Z}$. If the elements $e_1, \ldots, e_r$ generate $\operatorname{im}\alpha$, then any element $a \in \tilde{A}$ can be written as follows:

$$a = (z\bar{z})^{-n}(\lambda_1 e_1 + \cdots + \lambda_r e_r),$$

where $\lambda_i \in \mathbf{Z}$. This proves (a) $\Rightarrow$ (b).

The implication (b) $\Rightarrow$ (c) follows from the equalities

$$z^{-1} = 1 - t, \qquad \bar{z}^{-1} = 1 - t^{-1},$$

which show that any polynomial of z^{-1} and $\bar{z}^{-1}$ can be expressed as a polynomial of t and t^{-1}.

The assertion (c) $\Rightarrow$ (d) was proved in the proof of Corollary (1.3) in **[12]**.

Let us now prove that (d) $\Rightarrow$ (a). Let $\tilde{A}$ be a finitely-generated Λ-module and $\Delta(t)$ some integer polynomial such that $\Delta\tilde{A} = 0$ and $\Delta(1) = 1$. The last condition implies that

$$\Delta(t) = c_0(1-t)^n + c_1(1-t)^{n-1} + \cdots + c_{n-1}(1-t) + 1.$$

For any $a \in \tilde{A}$,

$$c_0(1-t)^n a + c_1(1-t)^{n-1}a + \cdots + c_{n-1}(1-t)a + a = 0.$$

Thus it follows that

$$\left[-c_0(1-t)^{n-1} - c_1(1-t)^{n-2} - \cdots - c_{n-1}\right] \cdot (1-t)a = a,$$

i.e. the multiplications by $(1-t)$ and by $-c_0(1-t)^{n-1} - \cdots - c_{n-1}$ are a pair of mutually inverse isomorphisms. So $\tilde{A}$ may be considered as an L-module. Then for all $a \in \tilde{A}$,

$$c_0 a + c_1 z a + \cdots + c_{n-1} z^{n-1} a + z^n a = 0$$

where $z = (1-t)^{-1}$.

Suppose the elements $e_1, e_2, \ldots, e_m \in \tilde{A}$ generate $\tilde{A}$ over Λ. Consider the following set:

$$\{e_1, ze_1, \ldots, z^{n-1}e_1, e_2, ze_2, \ldots, z^{n-1}e_2, \ldots, e_m, ze_m, \ldots, z^{n-1}e_m\}.$$

Let $A \subset \tilde{A}$ be the subgroup generated by this set. Then it has the following properties:

(1) A is finitely-generated over $\mathbf{Z}$;

(2) A is invariant under z and so it is a P-module;

(3) A generates $\tilde{A}$ over L (because its subset $e_1, \ldots, e_m$ does).

Consider the homomorphism $f\colon A \otimes_P L \to \tilde{A}$, where $f(a \otimes l) = la$, $l \in L$, $a \in A$. Any element of $A \otimes_P L$ may be presented in the form of $a \otimes (z\bar{z})^{-q}$, where $q \geqslant 0$, $a \in A$. If $f(a \otimes (z\bar{z})^{-q}) = 0$ then $(z\bar{z})^{-q}a = 0$ in $\tilde{A}$ and so $a = 0$. Then f is a monomorphism. On the other hand, condition (3) implies that f is an epimorphism. This completes the proof.

As a conclusion I should mention the papers **[5, 6]** where the questions concerning the topic of this section are considered.

6. Finite *P*-modules. In this section the canonical decomposition of a finite P-module into the sum of three of its submodules is studied and the relation between this construction and the functor $\otimes_P L$ from the preceding section is found.

Recall that $P = \mathbf{Z}[z]$, $L = \mathbf{Z}[z, z^{-1}, \bar{z}^{-1}]$, where $\bar{z} = 1 - z$.

6.1. Let A be a finite P-module. The subgroup $(z\bar{z})^n A$ does not depend on n for sufficiently large n. Denote this subgroup by $(A)_0$. Consider also the subgroups

$$(A)_+ = \{a \in A;\ \exists n \geqslant 0, z^n a = 0\},$$
$$(A)_- = \{a \in A;\ \exists n \geqslant 0, \bar{z}^n a = 0\}.$$

Clearly $(A)_0$, $(A)_+$, $(A)_-$ are submodules of A. Any homomorphism $f\colon A \to B$, of finite P-modules, maps $(A)_0$ into $(B)_0$. Really, if $a \in (A)_0$ then, for all n, a may be written in the form of $a = (z\bar{z})^n a_1$, where $a_1 \in A$. Thus $f(a) = (z\bar{z})^n f(a_1)$ and so $f(a) \in (B)_0$. The restriction of f on $(A)_0$ will be denoted by $f_0\colon (A)_0 \to (B)_0$.

Similarly, f maps $(A)_+$ into $(B)_+$, and $(A)_-$ into $(B)_-$. Its restrictions will be denoted by $f_+ \colon (A)_+ \to (B)_+$ and $f_-\colon (A)_- \to (B)_-$, respectively.

6.2. PROPOSITION. I. *For any finite P-module A the module $(A)_0$ admits an L-module structure, the module $(A)_+$ admits a $\mathbf{Z}[z, \bar{z}^{-1}]$-module structure, and the module $(A)_-$ admits a $\mathbf{Z}[z, z^{-1}]$-module structure.*

II. *Any finite P-module A is isomorphic over P to the direct sum $(A)_0 \oplus (A)_+ \oplus (A)_-$.*

III. *For any finite P-module A there is an L-module isomorphism $A \otimes_P L \approx (A)_0$.*

IV. *A sequence of finite P-modules*

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact if and only if the following three sequences are exact:

$$(A)_0 \xrightarrow{f_0} (B)_0 \xrightarrow{g_0} (C)_0, \qquad (A)_+ \xrightarrow{f_+} (B)_+ \xrightarrow{g_+} (C)_+,$$

$$(A)_- \xrightarrow{f_-} (B)_- \xrightarrow{g_-} (C)_-.$$

PROOF OF STATEMENT I. From the finiteness of A it follows that there exists an integer $n \geqslant 0$ such that $(A)_0 = (z\bar{z})^m A$ for all $m \geqslant n$. Then $(z\bar{z})(A)_0 = (A)_0$. Therefore the multiplication by $z\bar{z} \in P$ is an epimorphism $(A)_0 \to (A)_0$ and hence an isomorphism. This proves the fact that $(A)_0$ admits an L-module structure.

To prove that $(A)_+$ admits a $\mathbf{Z}[z, \bar{z}^{-1}]$-module structure note that there is an integer $m \geqslant 0$ such that $z^m (A)_+ = 0$. Then for all $a \in (A)_+$ the following equality holds:

$$\left(1 + z + \cdots + z^{m-1}\right) \cdot \bar{z} \cdot a = a,$$

which means that the multiplication by $\bar{z}$ is an isomorphism $(A)_+ \to (A)_+$. This proves our statement.

One can prove that $(A)_-$ admits a $\mathbf{Z}[z, z^{-1}]$-module structure similarly.

PROOF OF STATEMENT II. According to statement I, the multiplication by $\bar{z}$ is an isomorphism $(A)_+ \to (A)_+$. Therefore there is an N such that $(\bar{z})^N a = a$ for all $a \in (A)_+$. If $a \in (A)_+ \cap (A)_-$ then for sufficiently large k we shall have $a = (\bar{z})^{kN} a = 0$. So $(A)_+ \cap (A)_- = 0$.

Let $K = \{a \in A;\ \exists n \geqslant 0,\ (z\bar{z})^n a = 0\}$. Let us define the homomorphisms $\tau_+ \colon K \to K$ and $\tau_-\colon K \to K$. To do this, note that for $a \in K$, $(z\bar{z})^n a = 0$ for some $n \geqslant 0$ and thus $z^n a \in (A)_-$ and $\bar{z}^n a \in (A)_+$. By virtue of statement I there is a unique element $a_- \in (A)_-$ such that $z^n a_- = z^n a$. Similarly, there is a unique element $a_+ \in (A)_+$ such that $\bar{z}^n a_+ = \bar{z}^n a$. Put $\tau_+(a) = a_+$, $\tau_-(a) = a_-$. It is easy to see that τ_+ and τ_- are well defined and do not depend on the choice of n.

The homomorphisms τ_+ and τ_- are projectors: $\tau_+^2 = \tau_+$, $\tau_-^2 = \tau_-$. Besides, $\tau_+ \circ \tau_- = 0 = \tau_- \circ \tau_+$. Let us show that $\tau_+ + \tau_- = 1: K \to K$. If $a \in K$ then $a_+ = \tau_+(a) \in (A)_+$, $a_- = \tau_-(a) \in (A)_-$ and $z^n a = z^n a_-$, $\bar{z}^n a = \bar{z}^n a_+$ for some $n \geq 0$. There is an $m \geq n$ such that $z^m a_+ = 0 = \bar{z}^m a_-$. Then

$$z^m(a - a_+ - a_-) = 0 \quad \text{and} \quad \bar{z}^m(a - a_+ - a_-) = 0.$$

Therefore $a - a_+ - a_- \in (A)_+ \cap (A)_- = 0$ and so $a = a_+ + a_-$. This means that $\tau_+ + \tau_- = 1_K$.

From the just described relations between τ_+ and τ_- it follows that $K = \operatorname{im} \tau_+ \oplus \operatorname{im} \tau_-$ and $\operatorname{im} \tau_+ = \ker \tau_-$, $\operatorname{im} \tau_- = \ker \tau_+$. But clearly, $\ker \tau_- = (A)_+$, $\ker \tau_+ = (A)_-$. These prove that $K = (A)_+ \oplus (A)_-$.

Evidently, $K \cap (A)_0 = 0$. If $a \in A$ then $(z\bar{z})^n a \in (A)_0$ for some sufficiently large n and since $(A)_0$ admits an L-module structure, there is an $a_0 \in (A)_0$ such that $(z\bar{z})^n a = (z\bar{z})^n a_0$. Define the projector $\tau_0: A \to A$ by $\tau_0(a) = a_0$. Then $\operatorname{im} \tau_0 = (A)_0$ and $\ker \tau_0 = K$. From these it follows that $A = (A)_0 \oplus K$ and thus $A = (A)_0 \oplus (A)_+ \oplus (A)_-$. This completes the proof.

Proof of statement III. It follows directly from II and from the relations

$$(A)_0 \otimes_P L \approx (A)_0, \qquad (A)_+ \otimes_P L = 0, \qquad (A)_- \otimes_P L = 0.$$

The first of these ones follows from the fact that $(A)_0$ admits an L-module structure. To prove the second one, note that for $a \in (A)_+$, $l \in L$ we have $a \otimes l = z^n a \otimes z^{-n} l = 0$ when n is sufficiently large. The third one is similar.

Proof of statement IV. If the sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is exact and $b \in (B)_0$, $g(b) = 0$ then $b = f(a)$, where $a \in A$. But then $\tau_0(a) \in (A)_0$ and $f(\tau_0(a)) = \tau_0 f(a) = \tau_0 b = b$, where τ_0 is the projector constructed in the proof of statement II. This proves exactness of the sequence

$$(A)_0 \xrightarrow{f_0} (B)_0 \xrightarrow{g_0} (C)_0.$$

Exactness of two other sequences may be proved similarly. The inverse assertion follows from II.

6.3. Proposition. *Let A be a finitely-generated over $\mathbf{Z}$ P-module and $\tilde{A} = A \otimes_P L$. Let $\nu: A \to \tilde{A}$ be the homomorphism acting by the formula $\nu(a) = a \otimes 1$. Then the kernel of the restriction of ν on $T(A)$ is equal to $(T(A))_+ \oplus (T(A))_-$ and the restriction of ν on $(T(A))_0$ is an isomorphism $(T(A))_0 \to_{\approx} T(\tilde{A})$.*

Recall that $T(X)$ denotes the $\mathbf{Z}$-torsion subgroup of X.

The proof of Proposition 6.3 will be given below in subsection 6.6.

6.4. Corollary. *If $\tilde{A}$ is a module of type K then the group $T(\tilde{A})$ is finite.*

This statement was proved by M. Kervaire [7]. Here it follows from Proposition 6.3 and Theorem 5.3.

In the proof of Proposition 6.3 we shall use the following fact.

6.5. LEMMA. *If $A \to B \to C$ is an exact sequence of P-modules then the sequence $A \otimes_P L \to B \otimes_P L \to C \otimes_P L$ is exact.*

This follows from the interpretation of the functor $A \mapsto A \otimes_P L$ as the limit functor of the direct system

$$A \to A \to A \to \cdots,$$

where all homomorphisms are multiplications by $z\bar{z} \in P$ (see proof of Lemma 5.5) and the fact that any limit functor of a direct system preserves exactness.

6.6. PROOF OF PROPOSITION 6.3. If $a \in (T(A))_+$ then $a \otimes 1 = z^n a \otimes z^{-n} = 0$. This proves $(T(A))_+ \subset \ker \nu$. Similarly $(T(A))_- \subset \ker \nu$. On the other hand, Proposition 6.2 implies $(T(A))_0 \approx T(A) \otimes_P L$ and since the inclusion $T(A) \to A$ induces a monomorphism $T(A) \otimes_P L \to A \otimes_P L$ (by Lemma 6.5), it follows that the restriction of ν on $(T(A))_0$ is a monomorphism. Therefore, $\ker(\nu|_{T(A)}) = (T(A))_+ + (T(A))_-$.

Let $x \in \tilde{A}$ and suppose $x = a \otimes (z\bar{z})^{-n}$, where $a \in A$, $n \geqslant 0$. If x is an element of finite order then there is an $m > 0$ such that $(z\bar{z})^m a = a_1 \in T(A)$. Let $a_2 = \tau_0(a_1)$, where τ_0 is the projector constructed in Proposition 6.2.II. Then $a_2 \in (T(A))_0$ and $(z\bar{z})^k a_2 = (z\bar{z})^k a_1$ for some $k > 0$. By virtue of 6.2.I there exists a unique element $a_3 \in (T(A))_0$ such that $(z\bar{z})^{n+m} a_3 = a_2$. Then

$$\begin{aligned} x &= a \otimes (z\bar{z})^{-n} = a_1 \otimes (z\bar{z})^{-n-m} = (z\bar{z})^k a_1 \otimes (z\bar{z})^{-n-m-k} \\ &= a_2 \otimes (z\bar{z})^{-n-m} = a_3 \otimes 1 = \nu(a_3). \end{aligned}$$

This proves that $\nu|_{(T(A))_0}$ is an epimorphism $(T(A))_0 \to T(\tilde{A})$. It was proved above that this homomorphism is a monomorphism. This completes the proof.

7. L-quintets. In this section we shall define an analogy of the operation $\otimes_P L$ (which was studied in §5) on the set of P-quintets. As a result, each P-quintet will be assigned with a similar algebraic object called an L-quintet. The main theorem of this section states that thus we have obtained a one-to-one correspondence between the set of R-equivalent classes of P-quintets and the set of isomorphism classes of L-quintets. The proof of this theorem will use two auxiliary Theorems 7.5 and 7.6 which will be proved in further sections.

7.1. An *L-quintet* is defined as a collection (A, B, α, l, ψ) consisting of

L-modules A and B;

L-homomorphism $\alpha\colon A \otimes_{\mathbf{Z}} \mathbf{Z}_2 \to B$;

forms $l\colon T(A) \otimes_{\mathbf{Z}} T(A) \to \mathbf{Q}/\mathbf{Z}$,

$\psi\colon B \otimes_{\mathbf{Z}} B \to \mathbf{Z}_4$,

if the following conditions hold:

(a) A is a module of type K (i.e. it is finitely-generated over $\mathbf{Z}[z^{-1}, \bar{z}^{-1}]$).

(b) The following sequence is exact

$$0 \to A \otimes \mathbf{Z}_2 \xrightarrow{\alpha} B \xrightarrow{\beta} \mathrm{Hom}_{\mathbf{Z}}(A; \mathbf{Z}_2) \to 0$$

where $\beta(b)(a) = \psi(b \otimes \alpha(\pi(a)))$, $b \in B$, $a \in A$ and $\pi\colon A \to A \otimes \mathbf{Z}_2$ is the projection;

(c) The pairing l is nondegenerate.

(d) The pairings l and ψ are ε-symmetric.

(e) The composition

$$B \xrightarrow{\tilde{\gamma}} A \xrightarrow{\pi} A \otimes \mathbf{Z}_2 \xrightarrow{\alpha} B$$

coincides with multiplication by 2. Here for $b \in B$, $\tilde{\gamma}(b)$ is defined by the condition

$$\psi(b \otimes \alpha(\pi(a))) = l(\tilde{\gamma}(b) \otimes a) \quad \text{for all } a \in T(A).$$

(f) $l(za \otimes b) = l(a \otimes \bar{z}b)$, where $a, b \in T(A)$.

(g) $\psi(za \otimes b) = \psi(a \otimes \bar{z}b)$, where $a, b \in B$.

The number ε, which may be 1 or -1, will be called *parity* of the L-quintet.

Two L-quintets $(A_\nu, B_\nu, \alpha_\nu, l_\nu, \psi_\nu)$, $\nu = 1, 2$, are called *isomorphic* if there exist L-isomorphisms $\varphi\colon A_1 \to A_2$, $\xi\colon B_1 \to B_2$ such that

$$l_1 = l_2 \circ \left(\varphi|_{T(A_1)} \otimes \varphi|_{T(A_1)}\right), \qquad \psi_1 = \psi_2 \circ (\xi \otimes \xi),$$

and the diagram

$$\begin{array}{ccc} A_1 \otimes \mathbf{Z}_2 & \xrightarrow{\varphi \otimes 1} & A_2 \otimes \mathbf{Z}_2 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 \\ B_1 & \xrightarrow[\xi]{} & B_2 \end{array}$$

is commutative.

Note. We will further suppose that the group $\mathrm{Hom}_{\mathbf{Z}}(A; \mathbf{Z}_2)$ is provided with the following P-module structure: $(zf)(a) = f(\bar{z}a)$, where $f \in \mathrm{Hom}(A; \mathbf{Z}_2)$, $a \in A$. Then it follows from (g) that the homomorphism β defined in (b) would be a module homomorphism

7.2. Now we show that any P-quintet (A, B, α, l, ψ) defines some L-quintet $(\tilde{A}, \tilde{B}, \tilde{\alpha}, \tilde{l}, \tilde{\psi})$ of the same parity. Put $\tilde{A} = A \otimes_P L$, $\tilde{B} = B \otimes_P L$. Let us define $\tilde{\alpha}$, so that the following diagram:

$$\begin{array}{ccc} \tilde{A}\mathbf{Z}_2 & \xrightarrow{\tilde{\alpha}} & \tilde{B} \\ \downarrow & & \downarrow = \\ (A \otimes \mathbf{Z}_2) \otimes_P L & \xrightarrow{\alpha \otimes 1} & B \otimes_P L \end{array}$$

is commutative, where the left vertical isomorphism is the interchanging of the factors: $\tilde{A} \otimes \mathbf{Z}_2 = (A \otimes_P L) \otimes \mathbf{Z}_2 \approx (A \otimes \mathbf{Z}_2) \otimes_P L$. Now let us define the pairing $\tilde{l}\colon T(\tilde{A}) \otimes T(\tilde{A}) \to \mathbf{Q}/\mathbf{Z}$. By Proposition 6.3 there is a natural isomorphism $(T(A))_0 \to T(\tilde{A})$ which we denote by κ. Then for $x, y \in T(\tilde{A})$ we put

$$\tilde{l}(x \otimes y) = l(\kappa^{-1}(x) \otimes \kappa^{-1}(y)).$$

The pairing $\tilde{\psi}\colon \tilde{B} \otimes \tilde{B} \to \mathbf{Z}_4$ we define as the composition $\tilde{\psi} = \psi \circ (\lambda \otimes \lambda)$, where $\lambda\colon \tilde{B} \to (B)_0$ is an isomorphism, the existence of which is due to the Proposition 6.2.III.

7.3. *Claim. The constructed collection* $(\tilde{A}, \tilde{B}, \tilde{\alpha}, \tilde{l}, \tilde{\psi})$ *is an L-quintet.*

For the proof we must verify conditions (a)–(g) of subsection 7.1. Condition (a) is satisfied by virtue of Theorem 5.3. By Lemma 6.5 the exact sequence

$$0 \to A \otimes \mathbf{Z}_2 \xrightarrow{\alpha} B \xrightarrow{\beta} \operatorname{Hom}(A; \mathbf{Z}_2) \to 0$$

gives the exact sequence

$$0 \to (A \otimes \mathbf{Z}_2) \otimes_P L \to B \otimes_P L \to \operatorname{Hom}(A; \mathbf{Z}_2) \otimes_P L \to 0.$$

The module on the left is naturally isomorphic to $\tilde{A} \otimes \mathbf{Z}_2$ and the module on the right is naturally isomorphic to

$$\operatorname{Hom}(A; \mathbf{Z}_2) \otimes_P L = (\operatorname{Hom}(A; \mathbf{Z}_2))_0 = \operatorname{Hom}((A \otimes \mathbf{Z}_2)_0; \mathbf{Z}_2) \approx \operatorname{Hom}(\tilde{A}; \mathbf{Z}_2).$$

Here Proposition 6.2.III was used. Substituting these in the exact sequence written above we get the exact sequence

$$0 \to \tilde{A} \otimes \mathbf{Z}_2 \xrightarrow{\tilde{\alpha}} \tilde{B} \xrightarrow{\tilde{\beta}} \operatorname{Hom}(\tilde{A}; \mathbf{Z}_2) \to 0,$$

where $\tilde{\beta}$ is defined by $\tilde{\psi}$ as in (b) of subsection 7.1. This proves (b).

To prove (c) it is sufficient to establish that the restriction $l|_{(T(A))_0}$ is nondegenerate. For all $a \in (T(A))_0$ and $b \in T(A)$ the formula $l(a \otimes b) = l(a \otimes \tau_0(b))$ is true, where τ_0 is the projector constructed in the proof of Proposition 6.2.II. (Really, $(z\bar{z})^n b = (z\bar{z})^n \tau_0(b)$ for some n. Let $a_1 \in (T(A))_0$ be an element such that $(z\bar{z})^n a_1 = a$. Then

$$l(a \otimes b) = l\big(a_1 \otimes (z\bar{z})^n b\big) = l\big(a_1 \otimes (z\bar{z})^n \tau_0(b)\big) = l\big(a \otimes \tau_0(b)).\big)$$

It follows from this formula that if $l(a \otimes c) = 0$ for all $c \in (T(A))_0$ then $l(a \otimes b) = 0$ for all $b \in T(A)$ and, since l is nondegenerate, $a = 0$. This proves (c).

The properties (d), (f), (g) evidently follow from definitions.

Consider the homomorphism $\tilde{\gamma}\colon \tilde{B} \to \tilde{A}$ defined as described in (e) of 7.1. This homomorphism actually takes its values in $T(\tilde{A})$ and so the composition written in (e) of 7.1 coincides with

$$\tilde{B} \xrightarrow{\tilde{\gamma}} T(\tilde{A}) \xrightarrow{\pi} \tilde{A} \otimes \mathbf{Z}_2 \xrightarrow{\tilde{\alpha}} \tilde{B}$$

where all groups are finite. From Propositions 6.2.III and 6.3 it follows that the last composition coincides with

$$(B)_0 \xrightarrow{\gamma_0} (T(A))_0 \xrightarrow{\pi_0} (A \otimes \mathbf{Z}_2)_0 \xrightarrow{\alpha_0} (B)_0,$$

which is a restriction of the similar composition in the definition of P-quintet (see 4.1, property (e)). This implies (e).

Thus Claim 7.3 is proved.

The obtained L-quintet $(\tilde{A}, \tilde{B}, \tilde{\alpha}, \tilde{l}, \tilde{\psi})$ will be denoted by $(A, B, \alpha, l, \psi) \otimes_P L$.

The central result of this section is the following

7.4. THEOREM. I. *Any two P-quintets are R-equivalent if and only if the corresponding L-quintets are isomorphic.*

II. *For any L-quintet* $(\tilde{A}, \tilde{B}, \tilde{\alpha}, \tilde{l}, \tilde{\psi})$ *there exists a P-quintet* (A, B, α, l, ψ) *such that* $(\tilde{A}, \tilde{B}, \tilde{\alpha}, \tilde{l}, \tilde{\psi}) = (A, B, \alpha, l, \psi) \otimes_P L$.

The proof of Theorem 7.4.I will use the following two theorems.

7.5. THEOREM. *Any two P-quintets* $(A_\nu, B_\nu, \alpha_\nu, l_\nu, \psi_\nu)$, $\nu = 1, 2$, *are R-equivalent if and only if there exist P-homorphisms* $\varphi\colon A_1 \to A_2$, $\xi\colon B_1 \to B_2$, $\hat{\varphi}\colon A_2 \to A_1$, $\hat{\xi}\colon B_2 \to B_1$ *such that the following conditions hold*:

(a) *the following diagrams are commutative*:

$$\begin{array}{ccccccc} A_1 \otimes \mathbf{Z}_2 & \xrightarrow{\varphi\otimes 1} & A_2 \otimes \mathbf{Z}_2 & \qquad & A_1 \otimes \mathbf{Z}_2 & \xleftarrow{\hat{\varphi}\otimes 1} & A_2 \otimes \mathbf{Z}_2 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 & & \downarrow \alpha_1 & & \downarrow \alpha_2 \\ B_1 & \xrightarrow[\xi]{} & B_2 & & B_1 & \xleftarrow[\hat{\xi}]{} & B_2 \end{array}$$

(b) $\psi_1 \circ (\hat{\xi} \otimes 1_{B_1}) = \psi_2 \circ (1_{B_2} \otimes \xi)$,

(c) $l_1 \circ (\hat{\varphi}\,|_{T(A_2)} \otimes 1_{T(A_1)}) = l_2 \circ (1_{T(A_2)} \otimes \varphi\,|_{T(A_1)})$,

(d) *the homomorphisms* $\varphi \circ \hat{\varphi}$, $\hat{\varphi} \circ \varphi$, $\xi \circ \hat{\xi}$, $\hat{\xi} \circ \xi$ *coincide with multiplications by* $(z\bar{z})^m \in P$ *for some* $m \geqslant 0$.

A P-quintet (A, B, α, l, ψ) will be called *minimal* if the module A is minimal.

7.6. THEOREM. *Any P-quintet is R-equivalent to some minimal P-quintet.*

The proofs of Theorems 7.5 and 7.6 will be obtained in further sections.

7.7. PROOF OF THEOREM 7.4.I. Suppose $(A_\nu, B_\nu, \alpha_\nu, l_\nu, \psi_\nu)$, $\nu = 1, 2$, are two contiguous P-quintets and let $\varphi\colon A_1 \to A_2$, $\xi\colon B_1 \to B_2$, $\hat{\varphi}\colon A_2 \to A_1$, $\hat{\xi}\colon B_2 \to B_1$ be homomorphisms from the definition of contiguity of P-quintets (see 4.1). Let $(\tilde{A}_\nu, \tilde{B}_\nu, \tilde{\alpha}_\nu, \tilde{l}_\nu, \tilde{\psi}_\nu) = (A_\nu, B_\nu, \alpha_\nu, l_\nu, \psi_\nu) \otimes_P L$, $\nu = 1, 2$. Let us show that these L-quintets are isomorphic. Define the homomorphisms

$$\Phi\colon \tilde{A}_1 \to \tilde{A}_2, \qquad \Xi\colon \tilde{B}_1 \to \tilde{B}_2$$

by the formulas

$$\Phi(a \otimes 1) = \varphi(a) \otimes z^{-1}, \qquad \Xi(b \otimes 1) = \xi(b) \otimes z^{-1},$$

where $a \in A_1$, $b \in B_1$. Similarly, define

$$\hat{\Phi}\colon \tilde{A}_2 \to \tilde{A}_1, \qquad \hat{\Xi}\colon \tilde{B}_2 \to \tilde{B}_1$$

by the formulas

$$\hat{\Phi}(a \otimes 1) = \hat{\varphi}(a) \otimes \bar{z}^{-1}, \qquad \hat{\Xi}(b \otimes 1) = \hat{\xi}(b) \otimes \bar{z}^{-1}$$

for $a \in A_2$, $b \in B_2$. The condition (d) of the definition of P-quintet contiguity in subsection 4.1 implies that Φ and $\hat{\Phi}$ are mutually inverse isomorphisms. Similarly, Ξ and $\hat{\Xi}$ are mutually inverse isomorphisms. It is easy to see that Φ and Ξ satisfy all conditions of L-quintet isomorphism in subsection 7.1.

This proves that R-equivalent P-quintets define isomorphic L-quintets.

Now suppose $(A_\nu, B_\nu, \alpha_\nu, l_\nu, \psi_\nu)$, $\nu = 1, 2$, are two P-quintets of the same parity and such that their corresponding L-quintets $(\tilde{A}_\nu, \tilde{B}_\nu, \tilde{\alpha}_\nu, \tilde{l}_\nu, \tilde{\psi}_\nu)$, $\nu = 1, 2$, are isomorphic. We want to prove that the initial P-quintets are R-equivalent. By Theorem

7.6 we may additionally suppose that the initial P-quintets are minimal. Then, by Lemma 5.5, the homomorphisms μ_ν: $A_\nu \to A_\nu \otimes_P L = \tilde{A}_\nu$, where $\mu_\nu(a) = a \otimes 1$ for $a \in A_\nu$, $\nu = 1, 2$, are monomorphisms. Let $\tilde{\varphi}$: $\tilde{A}_1 \to \tilde{A}_2$, $\tilde{\xi}$: $\tilde{B}_1 \to \tilde{B}_2$ be isomorphisms from the definition of L-quintet isomorphism (see subsection 7.1). Arguments similar to those used in subsection 5.8 prove that there is an integer $n \geqslant 0$ such that

$$\tilde{\varphi}\left((z\bar{z})^n \operatorname{im} \mu_1\right) \subset \operatorname{im} \mu_2, \qquad (z\bar{z})^n \operatorname{im} \mu_2 \subset \tilde{\varphi}(\operatorname{im} \mu_1).$$

We may additionally suppose that n is so large that

$$z^n(A_\nu \otimes Z_2)_+ = 0 = \bar{z}^n(A_\nu \otimes Z_2)_-, \qquad z^n(B_\nu)_+ = 0 = \bar{z}^n(B_\nu)_-$$

for $\nu = 1, 2$. Consider the homomorphisms φ: $A_1 \to A_2$, $\hat{\varphi}$: $A_2 \to A_1$ given by the formulas

$$\varphi(a) = \mu_2^{-1} \circ \tilde{\varphi} \circ \mu_1\left((z\bar{z})^n a\right), \qquad a \in A_1,$$
$$\hat{\varphi}(a) = \mu_1^{-1} \circ \tilde{\varphi}^{-1} \circ \mu_2\left((z\bar{z})^n a\right), \qquad a \in A_2.$$

Consider also the homomorphisms λ_ν: $B_\nu \to \tilde{B}_\nu$ for which $\lambda_\nu(b) = b \otimes 1$, $b \in B_\nu$, $\nu = 1, 2$. These homomorphisms may not be monomorphisms. But according to 6.2 the homomorphisms $\bar{\lambda}_\nu = \lambda_\nu|_{(B_\nu)_0}$ are isomorphisms. Define ξ: $B_1 \to B_2$, $\hat{\xi}$: $B_2 \to B_1$ by

$$\xi(b) = (z\bar{z})^n \bar{\lambda}_2^{-1} \circ \tilde{\xi} \circ \lambda_1(b), \qquad b \in B_1,$$
$$\hat{\xi}(b) = (z\bar{z})^n \bar{\lambda}_1^{-1} \circ \tilde{\xi}^{-1} \circ \lambda_2(b), \qquad b \in B_2.$$

We omit the trivial verification that the homomorphisms φ, ξ, $\hat{\varphi}$, $\hat{\xi}$ satisfy all conditions of Theorem 7.5 (with $m = 2n$ in (d)). By this theorem the P-quintets $(A_\nu, B_\nu, l_\nu, \psi_\nu)$, $\nu = 1, 2$, are R-equivalent. Thus Theorem 7.4.I is proved.

7.8. Proof of Theorem 7.4.II. Let $(\tilde{A}, \tilde{B}, \tilde{\alpha}, \tilde{l}, \tilde{\psi})$ be an L-quintet. We want to construct a P-quintet (A, B, α, l, ψ) for which $(A, B, \alpha, l, \psi) \otimes_P L$ is isomorphic to $(\tilde{A}, \tilde{B}, \tilde{\alpha}, \tilde{l}, \tilde{\psi})$. By Theorem 5.3 there exists a finitely-generated over $\mathbf{Z}$, P-module A such that $A \otimes_P L \approx \tilde{A}$. By Theorem 5.2 and Lemma 5.6 we may suppose that A is minimal. Let the module B be given as the sum of

$$B = B_1 \oplus B_2 \oplus B_3 \oplus B_4 \oplus B_5$$

where

$$B_1 = \tilde{B}, \qquad B_2 = (A \otimes \mathbf{Z}_2)_+, \qquad B_3 = (A \otimes \mathbf{Z}_2)_-,$$
$$B_4 = (\operatorname{Hom}(A; \mathbf{Z}_2))_+, \qquad B_5 = (\operatorname{Hom}(A; \mathbf{Z}_2))_-.$$

Define α: $A \otimes \mathbf{Z}_2 \to B$ as the direct sum of the following three homomorphisms:

$$\alpha_0\colon (A \otimes \mathbf{Z}_2)_0 \underset{\approx}{\to} \tilde{A} \otimes \mathbf{Z}_2 \xrightarrow{\tilde{\alpha}} \tilde{B} = B_1 = (B)_0,$$

$$\alpha_+\colon (A \otimes \mathbf{Z}_2)_+ \underset{\approx}{\to} B_2 \hookrightarrow B, \qquad \alpha_-\colon (A \otimes \mathbf{Z}_2)_- \underset{\approx}{\to} B_3 \hookrightarrow B.$$

The not-denoted homomorphism in α_0 is the composition of the isomorphism $(A \otimes \mathbf{Z}_2)_0 \approx (A \otimes \mathbf{Z}_2) \otimes_P L$, which was constructed in Proposition 6.2.III, and of the isomorphism $(A \otimes \mathbf{Z}_2) \otimes_P L \approx (A \otimes_P L) \otimes \mathbf{Z}_2 = \tilde{A} \otimes \mathbf{Z}_2$, which exists owing to the commutativity of the tensor product operation. The not-denoted homomorphisms in the definitions of α_+ and α_- are the identity maps.

Now let us define form l: $T(A) \otimes T(A) \to \mathbf{Q}/\mathbf{Z}$. Let μ: $A \to \tilde{A}$ take $a \in A$ to $a \otimes 1 \in A \otimes_P L$. (Here we identify $A \otimes_P L$ with $\tilde{A}$.) Let $\kappa = \mu|_{T(A)}$. The minimality of A implies that μ is a monomorphism and κ is an isomorphism (see Lemma 5.5 and Proposition 6.3). Put $l = \tilde{l} \circ (\kappa \otimes \kappa)$.

Let us define form ψ: $B \otimes B \to \mathbf{Z}_4$ by the matrix

$$\begin{vmatrix} \tilde{\psi} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon\delta \\ 0 & 0 & 0 & \varepsilon\delta & 0 \\ 0 & 0 & \delta' & 0 & 0 \\ 0 & \delta' & 0 & 0 & 0 \end{vmatrix}$$

where the prime denotes the transposition and the forms

$$\delta\colon (A \otimes \mathbf{Z}_2)_+ \otimes (\mathrm{Hom}(A;\mathbf{Z}_2))_- \to \mathbf{Z}_2 \hookrightarrow \mathbf{Z}_4,$$
$$\sigma\colon (A \otimes \mathbf{Z}_2)_- \otimes (\mathrm{Hom}(A;\mathbf{Z}_2))_+ \to \mathbf{Z}_2 \hookrightarrow \mathbf{Z}_4$$

are restrictions of the canonical form

$$(A \otimes \mathbf{Z}_2) \otimes (\mathrm{Hom}(A;\mathbf{Z}_2)) \to \mathbf{Z}_2,$$

which takes $x \otimes f$ to $f(x)$ for $x \in A \otimes \mathbf{Z}_2$, $f \in \mathrm{Hom}(A;\mathbf{Z}_2)$. (Here we identify $\mathrm{Hom}(A;\mathbf{Z}_2)$ with $\mathrm{Hom}(A \otimes \mathbf{Z}_2;\mathbf{Z}_2)$.)

As the result of all these constructions we obtain the collection (A, B, α, l, ψ). One can easily see that it is a P-quintet, i.e. it satisfies conditions (a)–(g) from subsection 4.1. For example, to verify (e) notice that $\psi(b \otimes \alpha(\pi(a))) = 0$ for $a \in T(A)$ and $b \in (B)_+$ or $b \in (B)_-$ (it follows from the minimality of A). Therefore, the restriction of γ, which was defined in 4.1(e), on $(B)_+$ and on $(B)_-$ are equal to zero. Moreover, there is a commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\gamma} & T(A) \\ \eta\downarrow & & \approx\downarrow\kappa \\ \tilde{B} & \xrightarrow[\tilde{\gamma}]{} & T(A) \end{array}$$

where η is the projection on $B_1 = \tilde{B}$. This implies that the restriction of the composition

$$(*) \qquad B \xrightarrow{\gamma} T(A) \xrightarrow{\pi} A \otimes \mathbf{Z}_2 \xrightarrow{\alpha} B$$

on $(B)_+ \oplus (B)_-$ is equal to zero and its restriction on $(B)_0$ is equal to

$$\tilde{B} \xrightarrow{\tilde{\gamma}} T(\tilde{A}) \xrightarrow{\pi} \tilde{A} \otimes \mathbf{Z}_2 \xrightarrow{\tilde{\alpha}} \tilde{B} = B_1 \hookrightarrow B,$$

which coincides with multiplication by 2 by virtue of the definition of L-quintet. Since the groups $(B)_+$ and $(B)_-$ are of exponent 2, the whole composition $(*)$ coincides with the multiplication by 2.

This proves the property (e) of the definition of P-quintet in 4.1. Other properties (a)–(g) can be proved similarly. So (A, B, α, l, ψ) is a P-quintet. It is clear that

$$(A, B, \alpha, l, \psi) \otimes_P L = (\tilde{A}, \tilde{B}, \tilde{\alpha}, \tilde{l}, \tilde{\psi}).$$

Theorem 7.4 is proved.

8. Minimization of isometries. In this section Theorem 7.6 will be proved. Although this theorem is purely algebraic, it will be deduced by means of a general homotopy construction of isometries prompted by the geometrical construction of spherical modifications. This construction, also called modification, allows us to construct all isometries contiguous to the given one. We study the change of the homology modules under modifications and on this base prove that any isometry is R-equivalent to (X, u, z) for which all modules $H_i(X; \mathbf{Q})$ are minimal. Then we strengthen this theorem for isometries with length $X \leq 2$ and, using the relation between P-quintets and isometries (see 4.4 and 4.5), we obtain the proof of Theorem 7.6.

It is clear that the proof of Theorem 7.6 given here can be made purely algebraic by applying functors H_q, σ_{q+2} (see §§3, 4). However, the proof thus obtained would be more intricate. On the other hand, the technique of modifications, to the development of which the largest part of this section is devoted, can be applied as well to another problem on stable knots.

Let us begin with some easy facts concerning the category Stab_0. Recall that any diagram

$$A_1 \underset{\pi_1}{\overset{i_1}{\rightleftarrows}} B \underset{\pi_2}{\overset{i_2}{\leftrightarrows}} A_2$$

consisting of objects and morphisms of some additive category is called a *direct sum diagram* if the following relations hold:

$$\pi_1 \circ i_1 = 1_{A_1}, \qquad \pi_2 \circ i_2 = 1_{A_2}, \qquad i_1 \circ \pi_1 + i_2 \circ \pi_2 = 1_B.$$

From these equalities follows $\pi_1 \circ i_2 = 0 = \pi_2 \circ i_1$ (see [**13**, Chapter IX §1]). The object B is denoted $A_1 \oplus A_2$. Any two morphisms $f\colon C \to A_1$, $g\colon C \to A_2$ define morphism $f \oplus g\colon C \to A_1 \oplus A_2$ by the formula $f \oplus g = i_1 \circ f + i_2 \circ g$. Similarly, any two morphisms $\varphi\colon A_1 \to D$ and $\psi\colon A_2 \to D$ define morphism $(\varphi, \psi)\colon A_1 \oplus A_2 \to D$ by the formula $(\varphi, \psi) = \varphi \circ \pi_1 + \psi \circ \pi_2$.

8.1. PROPOSITION. *The diagram*

$$\mathcal{X}_1 \underset{\pi_1}{\overset{i_1}{\rightleftarrows}} \mathcal{Y}$$

consisting of the object and morphisms of the category Stab_0 *can be supplemented to a direct sum diagram if and only if* $\pi_1 \circ i_1 = 1_{\mathcal{X}_1}$. *For any two direct sum diagrams*

$$\mathcal{X}_1 \underset{\pi_1}{\overset{i_1}{\rightleftarrows}} \mathcal{Y} \underset{\pi_2}{\overset{i_2}{\leftrightarrows}} \mathcal{X}_2, \qquad \mathcal{X}_1 \underset{\pi_1}{\overset{i_1}{\rightleftarrows}} \mathcal{Y} \underset{\pi_3}{\overset{i_3}{\leftrightarrows}} \mathcal{X}_3,$$

there exists an S-equivalence $\varphi\colon \mathcal{X}_2 \to \mathcal{X}_3$ *with* $i_3 \circ \varphi = i_2$, $\varphi \circ \pi_2 = \pi_3$.

PROOF. Consider the S-map $f\colon \mathcal{Y} \to \mathcal{Y}$, $f = 1_{\mathcal{Y}} - i_1 \circ \pi_1$. From $\pi_1 \circ i_1 = 1_{\mathcal{X}_1}$ follows $f^2 = f$. Applying the splitting idempotents in Theorem 1.7, we obtain a virtual complex $\mathcal{X}_2$ and S-maps $i_2\colon \mathcal{X}_2 \to \mathcal{Y}$, $\pi_2\colon \mathcal{Y} \to \mathcal{X}_2$ with $\pi_2 \circ i_2 = 1_{\mathcal{X}_2}$ and $i_2 \circ \pi_2 = f = 1_{\mathcal{Y}} - i_1 \circ \pi_1$. Thus we get the direct sum diagram.

The uniqueness statement is evident: one may put $\varphi = \pi_3 \circ i_2$. The inverse to φ S-equivalence is $\psi = \pi_2 \circ i_3$.

8.2. We shall use the mapping-cone construction in the category Stab_0. If $f\colon \mathcal{X} \to \mathcal{Y}$ is an S-map then for some sufficiently large N the complexes $\Sigma^N\mathcal{X}$ and $\Sigma^N\mathcal{Y}$ are true and $\Sigma^N f$ may be realized by a continuous map $g\colon \Sigma^N\mathcal{X} \to \Sigma^N\mathcal{Y}$. The *mapping cone* $C(f)$ is defined to be the virtual complex $\Sigma^{-N}C(g)$, where $C(g)$ is the mapping cone of g in the usual sense.

For any S-map *the Puppe sequence* takes place,

$$\mathcal{X} \xrightarrow{f} \mathcal{Y}' \to C(f) \to \Sigma^1\mathcal{X} \xrightarrow{\Sigma^1 f} \Sigma^1\mathcal{Y},$$

being exact in the sense that it induces exact sequence of abelian groups in any homology and cohomology theory.

8.3. Further we shall use some well-known properties of the *stable Hurewicz homomorphism* $h_i\colon \sigma_i(\mathcal{X}) \to H_i(\mathcal{X})$:

(a) if $\mathcal{X}$ is a $(k-1)$-connected virtual complex then h_k is an isomorphism and h_{k+1} is an epimorphism,

(b) the rational Hurewicz homomorphism $\sigma_i(\mathcal{X}) \otimes \mathbf{Q} \to H_i(\mathcal{X}; \mathbf{Q})$ is an isomorphism for all $i \in \mathbf{Z}$.

8.4. Let us start to describe the modification construction. Suppose we are given an n-isometry $(\mathcal{X}, u, z)$ and a pair of S-maps

$$\alpha\colon \mathcal{X} \to \mathcal{L} \quad \text{and} \quad \beta\colon \mathcal{L} \to \mathcal{X}$$

such that $\beta \circ \alpha = z$. Here L is a virtual complex. Consider the virtual complex $\mathcal{M}$ which is dual to $\mathcal{L}$ by means of some duality $v\colon \mathcal{L} \otimes \mathcal{M} \to S^{n+1}$. Let us define S-maps $\gamma\colon \mathcal{X} \to \mathcal{M}$, $\delta\colon \mathcal{M} \to \mathcal{X}$ by demanding that the following relations (1), (2) hold:

$$v \circ (1_{\mathcal{L}} \otimes \gamma) = u \circ (\beta \otimes 1_{\mathcal{X}}), \tag{1}$$

$$v \circ (\alpha \otimes 1_{\mathcal{M}}) = u \circ (1_{\mathcal{X}} \otimes \delta). \tag{2}$$

These uniquely define γ and δ because u and v are dualities. We have $u \circ (1_{\mathcal{X}} \otimes \delta \circ \gamma) = v \circ (\alpha \otimes \gamma) = u \circ (\beta \circ \alpha \otimes 1_{\mathcal{X}}) = u \circ (\mathrm{iz} \otimes 1_{\mathcal{X}}) = u \circ (1_{\mathcal{X}} \otimes \bar{z})$ where $\bar{z} = 1 - z$ as usual. From this follows $\delta \circ \gamma = \bar{z}$ and so

$$\beta \circ \alpha + \delta \circ \gamma = 1_{\mathcal{X}}. \tag{3}$$

The S-maps α and γ define S-map $i\colon \mathcal{X} \to \mathcal{L} \oplus \mathcal{M}$, and the S-maps β and δ define S-map $\pi\colon \mathcal{L} \oplus \mathcal{M} \to \mathcal{X}$ (see the remark before subsection 8.1). Then (3) means that $\pi \circ i = 1_{\mathcal{X}}$. By Proposition 8.1 there exists a direct sum diagram

$$\mathcal{X} \underset{\pi}{\overset{i}{\rightleftarrows}} \mathcal{L} \oplus \mathcal{M} \underset{\pi_1}{\overset{i_1}{\leftrightarrows}} \mathcal{X}_1.$$

S-maps π_1 and i_1 give S-maps

$$\alpha_1 : \mathcal{X}_1 \to \mathcal{L}, \qquad \beta_1 : \mathcal{L} \to \mathcal{X}_1, \qquad \gamma_1 : \mathcal{X}_1 \to \mathcal{M}, \qquad \delta_1 : \mathcal{M} \to \mathcal{X}_1,$$

where α_1 and γ_1 define i_1, and β_1 and δ_1 define π_1 (in the sense of the remark before Proposition 8.1).

Let $z_1 : \mathcal{X}_1 \to \mathcal{X}_1$ be given by the formula

$$z_1 = \delta_1 \circ \gamma_1 \tag{4}$$

and $u_1 : \mathcal{X}_1 \otimes \mathcal{X}_1 \to S^{n+1}$ by

$$u_1 = -v \circ (\alpha_1 \otimes \gamma_1) + (-1)^n v' \circ (\gamma_1 \otimes \alpha_1). \tag{5}$$

We shall show in subsections 8.5–8.7 that the triplet $(\mathcal{X}_1, u_1, z_1)$ is an n-isometry contiguous to the initial one. To do this we have to obtain some relations between the S-maps constructed.

8.5. The equality $i \circ \pi + i_1 \circ \pi_1 = 1_{\mathcal{L} \oplus \mathcal{M}}$ is equivalent to the following four equalities:

$$\alpha \circ \beta + \alpha_1 \circ \beta_1 = 1_L, \tag{6}$$

$$\gamma \circ \beta + \gamma_1 \circ \beta_1 = 0, \tag{7}$$

$$\alpha \circ \delta + \alpha_1 \circ \delta_1 = 0, \tag{8}$$

$$\gamma \circ \delta + \gamma_1 \circ \delta_1 = 1_{\mathcal{M}}. \tag{9}$$

Similarly, equalities $\pi_1 \circ i_1 = 1_{\mathcal{X}_1}$, $\pi \circ i_1 = 0$, $\pi_1 \circ i = 0$ are equivalent to the following three equalities:

$$\beta_1 \circ \alpha_1 + \delta_1 \circ \gamma_1 = 1_{\mathcal{X}_1}, \tag{10}$$

$$\beta \circ \alpha_1 + \delta \circ \gamma_1 = 0, \tag{11}$$

$$\beta_1 \circ \alpha + \delta_1 \circ \gamma = 0. \tag{12}$$

8.6. Let us now verify that triplet $(\mathcal{X}_1, u_1, z_1)$ satisfies all conditions (a), (b), (c) of 2.1 and so it is an n-isometry.

Condition (b) is obviously satisfied. We have

$$u_1 \circ (z_1 \otimes 1_{\mathcal{X}_1}) = -v \circ (\alpha_1 \circ \delta_1 \circ \gamma_1 \otimes \gamma_1) + (-1)^n v' \circ (\gamma_1 \circ \delta_1 \circ \gamma_1 \otimes \alpha_1).$$

From (8) and (11) it follows that $\alpha_1 \circ \delta_1 \circ \gamma_1 = -\alpha \circ \delta \circ \gamma_1 = \alpha \circ \beta \circ \alpha_1$, and (9) implies $\gamma_1 \circ \delta_1 \circ \gamma_1 = \gamma_1 - \gamma \circ \delta \circ \gamma_1$. Then

$$u_1 \circ (z_1 \otimes 1_{\mathcal{X}_1}) = -v \circ (\alpha \circ \beta \circ \alpha_1 \otimes \gamma_1) + (-1)^n v' \circ (\gamma_1 \otimes \alpha_1) - (-1)^n v' \circ (\gamma \circ \delta \circ \gamma_1 \otimes \alpha_1).$$

Using (1) and (2) we obtain $v \circ (\alpha \circ \beta \circ \alpha_1 \otimes \gamma_1) = u \circ (\beta \circ \alpha_1 \otimes \delta \circ \gamma_1) = v \circ (\alpha_1 \otimes \gamma \circ \delta \circ \gamma_1)$. Consequently,

$$\begin{aligned} u_1 \circ (z_1 \otimes 1_{\mathcal{X}_1}) &= -v \circ (\alpha_1 \otimes \gamma \circ \delta \circ \gamma_1) + (-1)^n v' \circ (\gamma_1 \otimes \alpha_1) \\ &\quad - (-1)^n v' \circ (\gamma_1 \otimes \alpha \circ \beta \circ \alpha_1) \\ &= -v \circ (\alpha_1 \otimes \gamma \circ \delta \circ \gamma_1) + (-1)^n v' \circ (\gamma_1 \otimes \alpha_1 \circ \beta_1 \circ \alpha_1). \end{aligned}$$

From (11) and (7) it follows that $\gamma \circ \delta \circ \gamma_1 = \gamma_1 \circ \beta_1 \circ \alpha_1$, and so

$$u_1 \circ (z_1 \otimes 1_{\mathcal{X}_1}) = -v \circ (\alpha_1 \otimes \gamma_1 \circ \beta_1 \circ \alpha_1) + (-1)^n v' \circ (\gamma_1 \otimes \alpha_1 \circ \beta_1 \circ \alpha_1)$$
$$= u_1 \circ (1_{\mathcal{X}_1} \otimes \beta_1 \circ \alpha_1) = u_1 \circ (1_{\mathcal{X}_1} \otimes \bar{z}_1).$$

This proves (c) of subsection 2.1.

To prove (a) let us consider the n-isometry $(\mathcal{L} \oplus \mathcal{M}, U, Z)$, where $U: (\mathcal{L} \oplus \mathcal{M}) \otimes (\mathcal{L} \oplus \mathcal{M}) \to S^{n+1}$ and $Z: \mathcal{L} \oplus \mathcal{M} \to \mathcal{L} \oplus \mathcal{M}$ are given by the matrices

$$\begin{vmatrix} 0 & v \\ (-1)^{n+1} v' & 0 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 0 & 0 \\ 0 & 1_{\mathcal{M}} \end{vmatrix},$$

respectively. We have

$$U \circ (i \otimes i) = v \circ (\alpha \otimes \gamma) + (-1)^{n+1} v' \circ (\gamma \otimes \alpha)$$
$$= u \circ (\beta \circ \alpha \otimes 1_{\mathcal{X}}) + (-1)^{n+1} [u \circ (\beta \circ \alpha \otimes 1_{\mathcal{X}})]'$$
$$= u \circ (z \otimes 1_{\mathcal{X}}) + u \circ (1_{\mathcal{X}} \otimes z) = u.$$

Similarly, one can obtain $U \circ (i \otimes i_1) = 0$, $U \circ (i_1 \otimes i) = 0$, $U \circ (i_1 \otimes i_1) = -u_1$. Thus if $\zeta: \mathcal{X} \oplus \mathcal{X}_1 \to \mathcal{L} \oplus \mathcal{M}$ is an S-map defined by i and i_1 (in the sense of the remark before Proposition 8.1), then $U \circ (\zeta \otimes \zeta)$ is given by the matrix $\left|\begin{smallmatrix} u & 0 \\ 0 & -u_1 \end{smallmatrix}\right|$. On the other hand, U is a duality map and ζ is an S-equivalence (because i and i_1 take part in the direct sum diagram). Thus $U \circ (\zeta \otimes \zeta)$ is a duality and so is u_1.

Thus we have proved that $(\mathcal{X}_1, u_1, z_1)$ is an n-isometry.

8.7. Let us show that this n-isometry is contiguous to $(\mathcal{X}, u, z)$. Define S-maps φ: $\mathcal{X} \to \mathcal{X}_1$ and ψ: $\mathcal{X}_1 \to \mathcal{X}$ by the formulas $\varphi = \delta_1 \circ \gamma$, $\psi = -\beta \circ \alpha_1$. We have $\zeta \circ z = \delta_1 \circ \gamma \circ \beta \circ \alpha = -\delta_1 \circ \gamma_1 \circ \beta_1 \circ \alpha = \delta_1 \circ \gamma_1 \circ \delta_1 \circ \gamma = z_1 \circ \varphi$ (here equalities (7) and (12) have been used), $\psi \circ z_1 = -\beta \circ \alpha_1 \circ \delta_1 \circ \gamma_1 = \beta \circ \alpha \circ \delta \circ \gamma_1 = -\beta \circ \alpha \circ \beta \circ \alpha_1 = z \circ \psi$ (here equalities (8), (11) have been used). Besides, $\varphi \circ \psi = -\delta_1 \circ \gamma \circ \beta \circ \alpha_1 = \delta_1 \circ \gamma_1 \circ \beta_1 \circ \alpha_1 = \delta_1 \circ \gamma_1 - \delta_1 \circ \gamma_1 \circ \delta_1 \circ \gamma_1 = z_1 - z_1^2 = z_1 \circ \bar{z}_1$ ((7) and (10) have been used) and similarly, $\psi \circ \varphi = z \circ \bar{z}$. This proves properties (II) and (III) of the definition of contiguity in 2.2. We now have to verify (I). We obtain

$$u_1 \circ (1_{\mathcal{X}_1} \otimes \varphi) = -v \circ (\alpha_1 \otimes \gamma_1 \circ \delta_1 \circ \gamma) + (-1)^n v' \circ (\gamma_1 \otimes \alpha_1 \circ \delta_1 \circ \gamma)$$
$$= -v \circ (\alpha_1 \otimes \gamma) + v \circ (\alpha_1 \otimes \gamma \circ \delta \circ \gamma) + (-1)^{n+1} [v \circ (\alpha \circ \delta \circ \gamma \otimes \gamma_1)]'$$
$$= -u \circ (\beta \circ \alpha_1 \otimes 1_{\mathcal{X}}) + v \circ (\alpha_1 \otimes \gamma \circ \delta \circ \gamma) + (-1)^{n+1} [u \circ (\delta \circ \gamma \otimes \delta \circ \gamma_1)]'$$
$$= u \circ (\psi \otimes 1_{\mathcal{X}}) + u \circ (\beta \circ \alpha_1 \otimes \delta \circ \gamma) + u \circ (\delta \circ \gamma_1 \otimes \delta \circ \gamma) = u \circ (\psi \otimes 1_{\mathcal{X}}).$$

Here equalities (9), (8), (1) and (11) have been successively used.

8.8. So, if an n-isometry $(\mathcal{X}, u, z)$ and a pair of S-maps α: $\mathcal{X} \to \mathcal{L}$ and β: $\mathcal{L} \to \mathcal{X}$, such that $\beta \circ \alpha = z$, are given, the new n-isometry $(\mathcal{X}_1, u_1, z_1)$ is defined. We shall refer to it as the *modification* of the initial isometry corresponding to α and β. Any modification is contiguous to the initial isometry. The converse statement is also true: any isometry which is contiguous to $(\mathcal{X}, u, z)$ is its modification corresponding to some $\mathcal{L}$, α, β. We shall not use this fact and so shall not prove it here.

8.9. From (1) and (2) of subsection 8.4. it follows that to give α and β is the same as to give γ and δ. So in the situation when an n-isometry $(\mathcal{X}, u, z)$, a virtual complex M and S-maps $\gamma\colon \mathcal{X} \to \mathcal{M}$, $\delta\colon \mathcal{M} \to \mathcal{X}$ with $\delta \circ \gamma = \bar{z}$ are given, the modification $(\mathcal{X}_1, u_1, z_1)$ is well defined up to congruence.

8.10. Suppose we are given n-isometry $(\mathcal{X}, u, z)$. Consider the Puppe sequence

$$\mathcal{X} \xrightarrow{z} \mathcal{X} \to C(z) \xrightarrow{\Delta} \Sigma^1\mathcal{X} \xrightarrow{\Sigma^1 z} \Sigma^1\mathcal{X}$$

of the S-map z. Let us show that any S-map $f\colon \mathcal{Y} \to C(z)$ defines some virtual complex $\mathfrak{L}$ and a pair of S-maps $\alpha\colon \mathcal{X} \to \mathfrak{L}$, $\beta\colon \mathfrak{L} \to \mathcal{X}$ with $\beta \circ \alpha = z$. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "\Delta \circ f"] \arrow[d, "f"'] & \Sigma^1\mathcal{X} \arrow[r, "\kappa"] \arrow[d, "1"] & C(\Delta \circ f) \arrow[d, dashed, "\mu"] \\
C(z) \arrow[r, "\Delta"] & \Sigma^1\mathcal{X} \arrow[r, "\Sigma^1 z"] & \Sigma^1\mathcal{X}
\end{tikzcd}$$

where the upper horizontal line is the part of the Puppe sequence of S-map $\Delta \circ f$. Thus it follows that there exists an S-map μ, shown by the dashed arrow on the diagram and such that $\mu \circ \kappa = \Sigma^1 z$. Then we can put

$$\mathfrak{L} = \Sigma^{-1}C(\Delta \circ f), \qquad \alpha = \Sigma^{-1}\kappa, \qquad \beta = \Sigma^{-1}\mu.$$

Thus, if an n-isometry $(\mathcal{X}, u, z)$ is given, then any S-map $f\colon \mathcal{Y} \to C(z)$ defines some n-isometry $(\mathcal{X}_1, u_1, z_1)$, which is contiguous to the initial one and will be referred to as the *modification* of $(\mathcal{X}, u, z)$ corresponding to f.

Similarly, any S-map $g\colon \mathcal{Y} \to C(\bar{z})$ defines some virtual complex $\mathcal{M}$ and a pair of S-maps $\gamma\colon \mathcal{X} \to \mathcal{M}$, $\delta\colon \mathcal{M} \to \mathcal{X}$ with $\delta \circ \gamma = \bar{z}$. From the remark 8.9 it follows that any S-map $g\colon \mathcal{Y} \to C(\bar{z})$ defines an n-isometry contiguous to the initial one. It will also be referred to as the *modification* of the given isometry corresponding to g.

8.11. We shall further use the construction described in the preceding subsection when $\mathcal{Y}$ is a sphere.

The aim of the statements below is to establish the relation between the homology of the initial isometry and the homology of its modifications. Here we make some remarks necessary for the formulations of these statements.

We shall constantly bear in mind that for any isometry $(\mathcal{X}, u, z)$ and for any homology theory h_* the groups $h_i(\mathcal{X})$ have the natural structure of modules over the ring $P = \mathbf{Z}[z]$. This structure is given by the formula $za = z_*a$, where $a \in h_i(\mathcal{X})$.

The symbol $\Delta\colon C(z) \to \Sigma^1\mathcal{X}$ will denote the canonical S-map from the Puppe sequence at the beginning of 8.10. If $\xi \in \sigma_m(C(z))$ then $\partial\xi$ will denote the homology class in $H_{m-1}(\mathcal{X})$ which is the image of $\Delta_*(\xi) \in \sigma_m(\Sigma^1\mathcal{X})$ under the Hurewicz homomorphism.

8.12. Proposition. *Let $(\mathcal{X}, u, z)$ be any n-isometry and $\xi \in \sigma_m(C(z))$ be an element such that $\partial\xi \in H_{m-1}(\mathcal{X})$ has infinite order. Let $(\mathcal{X}_1, u_1, z_1)$ be the result of the modification of $(\mathcal{X}, u, z)$ corresponding to ξ. Then*

(a) rank $H_k(\mathcal{X}_1) =$ rank $H_k(\mathcal{X})$ *for $k \neq m - 1$, $k \neq n + 2 - m$;*

(b) *if $2m \neq n + 3$, then*

$$\operatorname{rank} H_{m-1}(\mathcal{X}_1) = \operatorname{rank} H_{m-1}(\mathcal{X}) - 1,$$
$$\operatorname{rank} H_{n+2-m}(\mathcal{X}_1) = \operatorname{rank} H_{n+2-m}(\mathcal{X}) - 1;$$

(c) *if* $2m = n + 3$, *then*

$$\operatorname{rank} H_{m-1}(\mathcal{X}_1) = \operatorname{rank} H_{m-1}(\mathcal{X}) - 2.$$

PROOF. The element $\xi \in \sigma_m(C(z))$ represents the S-map $\xi: S^m \to C(z)$. (Note that m can be negative here. For negative m, sphere S^m is defined to be the virtual complex (S^0, m).) As it was pointed out in 8.10, the S-map ξ defines some virtual complex L and S-maps $\alpha: \mathcal{X} \to \mathcal{L}$, $\beta: \mathcal{L} \to \mathcal{X}$ with $\beta \circ \alpha = z$. Then there is the Puppe sequence

$$\mathcal{X} \xrightarrow{\alpha} \mathcal{L} \to S^m \xrightarrow{\Delta \circ \xi} \Sigma^1 \mathcal{X}$$

where the S-map on the right represents the class $\partial\xi \in H_{m-1}(\mathcal{X})$. Since $\partial\xi$ has infinite order, the homomorphism

$$\alpha_*: H_i(\mathcal{X}; \mathbf{Q}) \to H_i(\mathcal{L}; \mathbf{Q})$$

is an isomorphism for $i \neq m - 1$ and an epimorphism for $i = m - 1$. In this last case the kernel of α_* is the one-dimensional subspace of $H_{m-1}(\mathcal{X}; \mathbf{Q})$ generated by $\partial\xi$. From this it follows that

(13) $$\operatorname{rank} H_i(\mathcal{X}) = \operatorname{rank} H_i(\mathcal{L}) \quad \text{for } i \neq m - 1,$$

(14) $$\operatorname{rank} H_{m-1}(\mathcal{X}) = \operatorname{rank} H_{m-1}(\mathcal{L}) + 1.$$

As it was pointed out in subsection 8.4, the collection $\mathcal{L}$, α, β defines a direct sum diagram

$$\mathcal{X} \underset{\pi}{\overset{i}{\rightleftarrows}} \mathcal{L} \oplus \mathcal{M} \underset{\pi_1}{\overset{i_1}{\leftrightarrows}} \mathcal{X}_1$$

and so

(15) $$\operatorname{rank} H_i(\mathcal{X}_1) = \operatorname{rank} H_i(\mathcal{L}) + \operatorname{rank} H_i(\mathcal{M}) - \operatorname{rank} H_i(\mathcal{X}).$$

Since there is the duality map $v: \mathcal{L} \otimes \mathcal{M} \to S^{n+1}$ then

(16) $$\operatorname{rank} H_i(\mathcal{M}) = \operatorname{rank} H_{n+1-i}(\mathcal{L})$$

for all i. Similarly,

$$\operatorname{rank} H_i(\mathcal{X}) = \operatorname{rank} H_{n+1-i}(\mathcal{X}).$$

From this and (13)–(16), all statements (a), (b), (c) of Proposition 8.12 easily follow.

Note that the analogy of Proposition 8.12 is true, where $C(\bar{z})$ appears instead of $C(z)$.

An n-isometry $(\mathcal{X}, u, z)$ will be called *rationally minimal* if P-modules $H_i(\mathcal{X}; \mathbf{Q})$ are minimal for all $i \in \mathbf{Z}$. The following theorem is a generalization of a well-known Trotter theorem **[14]** stating that any Seifert matrix is S-equivalent to some nondegenerate one.

8.13. THEOREM. *Any n-isometry is R-equivalent to some rationally minimal n-isometry.*

PROOF. If the n-isometry $(\mathcal{X}, u, z)$ is not rationally minimal then there is an element $a \in H_i(\mathcal{X}; \mathbf{Q})$, $a \neq 0$, such that $za = 0$ or $\bar{z}a = 0$. Suppose $za = 0$ for definiteness. From the exact sequence

$$H_{i+1}(C(z); \mathbf{Q}) \xrightarrow{\Delta_*} H_i(\mathcal{X}; \mathbf{Q}) \xrightarrow{z} H_i(\mathcal{X}; \mathbf{Q})$$

it follows that there exists $\eta \in H_{i+1}(C(z); \mathbf{Q})$ with $\Delta_* \eta = a$. Since the rational Hurewicz homomorphism is an isomorphism, there is a $\xi \in \sigma_{i+1}(C(z))$ which goes to $m\eta$ under the rational Hurewicz homomorphism, where $m \neq 0$, $m \in \mathbf{Z}$. Then using 8.11 we have $\partial\xi = ma$, and so $\partial\xi$ is of infinite order. Let $(\mathcal{X}_1, u_1, z_1)$ be the modification of $(\mathcal{X}, u, z)$ corresponding to ξ (see 8.10). Then by Proposition 8.12

$$\sum_{j=-\infty}^{\infty} \operatorname{rank} H_j(\mathcal{X}_1) = \sum_{j=-\infty}^{\infty} \operatorname{rank} H_j(\mathcal{X}) - 2.$$

The sums on each side of this equality are, in fact, finite. If the obtained isometry $(\mathcal{X}_1, u_1, z_1)$ is not rationally minimal, too, then one can apply to it the just described procedure to obtain $(\mathcal{X}_2, u_2, z_2)$ which is contiguous to $(\mathcal{X}_1, u_1, z_1)$ and

$$\sum_{j=-\infty}^{\infty} \operatorname{rank} H_j(\mathcal{X}_2) = \sum_{j=-\infty}^{\infty} \operatorname{rank} H_j(\mathcal{X}_1) - 2.$$

Using the induction we get the sequence of n-isometries $(\mathcal{X}_k, u_k, z_k)$ such that (a) $(\mathcal{X}_0, u_0, z_0) = (\mathcal{X}, u, z)$; (b) the isometry number k is defined if and only if the isometry number $k-1$ is not rationally minimal; (c) isometries with the neighbouring numbers are contiguous; (d) the following equality holds:

$$\sum_{j=-\infty}^{\infty} \operatorname{rank} H_j(\mathcal{X}_k) = \sum_{j=-\infty}^{\infty} \operatorname{rank} H_j(\mathcal{X}_{k-1}) - 2.$$

From this equality it follows that the constructed sequence of isometries must be finite. It is clear that its last member is a rationally minimal isometry R-equivalent to the initial one.

The theorem is proved.

If the n-isometry $(\mathcal{X}, u, z)$ is rationally minimal, then for all $i \in \mathbf{Z}$ the multiplication by $z\bar{z} \in P$ is an isomorphism $H_i(\mathcal{X}; \mathbf{Q}) \to H_i(\mathcal{X}; \mathbf{Q})$. From this it follows that *if two n-isometries $(\mathcal{X}_\nu, u_\nu, z_\nu)$, $\nu = 1, 2$, are R-equivalent and each of them is rationally minimal then P-modules $H_i(\mathcal{X}_1; \mathbf{Q})$ and $H_i(\mathcal{X}_2; \mathbf{Q})$ are isomorphic for all i.* In fact, since these isometries are R-equivalent there exist S-maps $\varphi\colon \mathcal{X}_1 \to \mathcal{X}_2$, $\psi\colon \mathcal{X}_2 \to \mathcal{X}_1$ such that

$$u_2 \circ (1_{\mathcal{X}_2} \otimes \varphi) = u_1 \circ (\psi \otimes 1_{\mathcal{X}_1}),$$
$$z_2 \circ \varphi = \varphi \circ z_1, \qquad z_1 \circ \psi = \psi \circ z_2,$$
$$\varphi \circ \psi = (z_2 \circ \bar{z}_2)^m, \qquad \psi \circ \varphi = (z_1 \circ \bar{z}_1)^m$$

where $m > 0$ is some integer. From the last two relations it follows that φ and ψ induce isomorphism of rational homology modules.

We shall now strengthen Theorem 8.13 for the $2q$-isometries on complexes with length $\leqslant 2$.

8.14. THEOREM. *Any* $2q$*-isometry* $(\mathcal{X}, u, z)$ *with* length $\mathcal{X} \leqslant 2$ *is R-equivalent to the* $2q$*-isometry* $(\mathcal{X}_0, u_0, z_0)$ *such that* length $\mathcal{X}_0 \leqslant 2$ *and all P-modules* $H_i(\mathcal{X}_0)$ *are minimal*, $i \in \mathbf{Z}$.

Notice at once that since length $\mathcal{X} \leqslant 2$, $H_q(\mathcal{X})$ and $H_{q+1}(\mathcal{X})$ may only be nonzero. Besides, the duality map u defines the isomorphism $H_{q+1}(\mathcal{X}) \approx H^q(\mathcal{X}; \mathbf{Z}) = \operatorname{Hom}(H_q(\mathcal{X}); \mathbf{Z})$ and it is a module isomorphism if P-module structure is introduced in Hom as it was pointed out in the Note at the end of 7.1. Thus $H_{q+1}(\mathcal{X})$ is minimal if $H_q(\mathcal{X})$ is.

For the proof of Theorem 8.14 we shall need some auxiliary statements.

8.15. PROPOSITION. *Let* $(\mathcal{X}, u, z)$ *be an* n*-isometry and* $\xi \in \sigma_m(C(z))$ *an element. Let* $(\mathcal{X}_1, u_1, z_1)$ *be the modification of* $(\mathcal{X}, u, z)$ *corresponding to* ξ (*see* 8.10). *Then*

(a) *if* $\partial\xi \in H_{m-1}(\mathcal{X})$ (*defined in* 8.11) *is a primitive element, then* $H_i(\mathcal{X}_1) \approx H_i(\mathcal{X})$ *for all* i *besides* $i = m - 1$ *and* $i = n + 2 - m$;

(b) *if* $\partial\xi$ *is primitive and* $2m < n + 3$, *then*

$$H_{m-1}(\mathcal{X}_1) \approx H_{m-1}(\mathcal{X})/(\partial\xi);$$

(c) *if* $\partial\xi$ *is of finite order then* $H_i(\mathcal{X}_1) \approx H_i(\mathcal{X})$ *for all* i *besides* $i = m - 1$, $i = m$, $i = n + 1 - m$;

(d) *if* $\partial\xi$ *is of finite order and* $2m < n + 2$ *then*

$$H_{m-1}(\mathcal{X}_1) \approx H_{m-1}(\mathcal{X})/(\partial\xi);$$

(e) *if* $\partial\xi$ *is of finite order and* $2m = n + 2$ *then in* $H_{m-1}(\mathcal{X}_1)$ *there is an element* κ *of infinite order such that*

$$H_{m-1}(\mathcal{X}_1)/(\kappa) \approx H_{m-1}(\mathcal{X})/(\partial\xi).$$

This proposition remains true if one replaces $C(z)$ by $C(\bar{z})$.

Recall that an element a of a finitely-generated abelian group A is called *primitive* if there exists a homomorphism $\nu\colon A \to \mathbf{Z}$ with $\nu(a) = 1$.

The proof of the Proposition 8.15 will use the following lemma.

8.16. LEMMA. *Suppose that in an exact sequence of abelian groups*

$$0 \to A \xrightarrow{i} B \oplus C \xrightarrow{j} D \to 0,$$

i is the sum of homomorphisms $f\colon A \to B$ and $g\colon A \to C$. If f is an epimorphism then the restriction of j to C defines an isomorphism $C/g(\ker(f)) \xrightarrow{\approx} D$.

PROOF OF THE LEMMA. Let $k\colon C \to D$ be the restriction of j to C. If $d \in D$ then $d = j(b + c)$, for some $b \in B$ and $c \in C$. But $b = f(a)$, where $a \in A$. Then $j(c - g(a)) = d$ and $c - g(a) \in C$. Thus k is an epimorphism. The kernel of k clearly coincides with $g(\ker(f))$ and so the lemma follows.

8.17. PROOF OF PROPOSITION 8.15. The element $\xi \in \sigma_m(C(z))$ is an S-map $\xi\colon S^m \to C(z)$. This S-map defines virtual complex $\mathfrak{L}$ and S-maps $\alpha\colon \mathcal{X} \to \mathfrak{L}$, $\beta\colon \mathfrak{L} \to \mathcal{X}$ with $\beta \circ \alpha = z$ (see subsection 8.10). According to the construction, α takes part in the following Puppe sequences:

$$\mathcal{X} \xrightarrow{\alpha} \mathfrak{L} \to S^m \xrightarrow{\Delta \circ \xi} \Sigma^1 \mathcal{X}. \tag{17}$$

As was shown in subsection 8.4, the collection $\mathfrak{L}$, α, β defines the direct sum diagram

$$\mathfrak{X} \underset{\pi}{\overset{i}{\rightleftarrows}} \mathfrak{L} \oplus \mathfrak{M} \underset{\pi_1}{\overset{i_1}{\leftrightarrows}} \mathfrak{X}_1,$$

and so there is an exact sequence

$$0 \to H_i(\mathfrak{X}) \overset{i_*}{\to} H_i(\mathfrak{L}) \oplus H_i(\mathfrak{M}) \overset{\pi_{1*}}{\to} H_i(\mathfrak{X}_1) \to 0. \tag{18}$$

Here i_* is the sum of α_* and γ_* (we are using the notations of 8.4). Considering the sequence induced by (17), we see that if $\partial\xi$ has infinite order, the homomorphism α_*: $H_i(\mathfrak{X}) \to H_i(\mathfrak{L})$ is an isomorphism for $i \neq m - 1$ and is an epimorphism with kernel $(\partial\xi)$ for $i = m - 1$. Applying Lemma 8.16 to the exact sequence (18) we see that the restriction of π_{1*} to $H_i(\mathfrak{M})$ is an isomorphism for $i \neq m - 1$ and is an epimorphism for $i = m - 1$. Thus in the notations of subsection 8.4 we get that

$$\delta_{1*}\colon H_i(\mathfrak{M}) \to H_i(\mathfrak{X}_1) \tag{19}$$

is an isomorphism for $i \neq m - 1$ and an epimorphism with kernel $\gamma_*(\partial\xi)$ for $i = m - 1$.

The duality map $v\colon \mathfrak{L} \otimes \mathfrak{M} \to S^{n+1}$ (see 8.4) defines an isomorphism

$$D\colon H_i(\mathfrak{M}) \to H^{n+1-i}(\mathfrak{L}),$$

where $D(z) = v^* z_{n+1}/z$, $z \in H_i(\mathfrak{M})$ and $z_{n+1} \in H^{n+1}(S^{n+1})$ is the fundamental class. The duality map $u\colon \mathfrak{X} \otimes \mathfrak{X} \to S^{n+1}$ similarly defines an isomorphism

$$D_1\colon H_i(\mathfrak{X}) \to H^{n+1-i}(\mathfrak{X}).$$

The equality (2) in subsection 8.4 implies that the following diagram is commutative up to sign:

$$\begin{array}{ccc} H^{n+1-i}(\mathfrak{L}) & \overset{\alpha^*}{\to} & H_{n+1-i}(\mathfrak{X}) \\ \approx\uparrow D & & \approx\uparrow D_1 \\ H_i(\mathfrak{M}) & \overset{\delta_*}{\to} & H_i(\mathfrak{X}) \end{array}$$

Considering the sequence induced by (17) in cohomology we see that

$$\alpha^*\colon H^j(\mathfrak{L}) \to H^j(\mathfrak{X})$$

is an isomorphism for $j \neq m - 1$ (if we additionally suppose that $\partial\xi$ is a primitive element). This implies that the homomorphism

$$\delta_*\colon H_i(\mathfrak{M}) \to H_i(\mathfrak{X}) \tag{20}$$

is an isomorphism for $i \neq n + 2 - m$. Now comparing the obtained information on homomorphisms (19) and (20) we get (a).

If $2m < n + 3$, then $m - 1 < n + 2 - m$. Therefore δ_{1*}: $H_{m-1}(\mathfrak{M}) \to H_{m-1}(\mathfrak{X}_1)$ is an epimorphism with kernel $\gamma_*((\partial\xi))$ and δ_*: $H_{m-1}(\mathfrak{M}) \to H_{m-1}(\mathfrak{X})$ is an isomorphism. Then

$$\delta_{1*} \circ \delta_*^{-1}\colon H_{m-1}(\mathfrak{X}) \to H_{m-1}(\mathfrak{X}_1)$$

is an epimorphism with kernel $\delta_* \circ \gamma_*((\partial\xi))$. Bit in subsection 8.4 we have proved that $\delta \circ \gamma = \bar{z} = 1 - z$, and on the other hand $\Sigma^1 z \circ \Delta = 0$ (see 8.10). Thus $z(\partial\xi) = (\Sigma^1 z)_* \circ \Delta_*(\xi) = 0$, $\delta_* \circ \gamma_*((\partial\xi)) = (\partial\xi)$, and so $H_{m-1}(\mathfrak{X}_1)$ is isomorphic to $H_{m-1}(\mathfrak{X})/(\partial\xi)$. This proves (b).

Proofs of statements (c), (d), (e) will be obtained similarly. If $\partial\xi \in H_{m-1}(\mathfrak{X})$ is of finite order then

$$\alpha_*: H_i(\mathfrak{X}) \to H_i(\mathfrak{L})$$

is an isomorphism for $i \neq m - 1$, $i \neq m$, and α_* is an epimorphism with kernel $(\partial\xi)$ for $i = m - 1$. This information and Lemma 8.16 as above imply that the homomorphism

$$\delta_{1*}: H_i(\mathfrak{M}) \to H_i(\mathfrak{X}_1) \tag{21}$$

is an isomorphism for $i \neq m - 1$, $i \neq m$, and it is an epimorphism with kernel $\gamma_*((\partial\xi))$ for $i = m - 1$. On the other hand, $\alpha^*: H^j(\mathfrak{L}) \to H^j(\mathfrak{X})$ is an isomorphism for $j \neq m$. Using the duality maps u and v one deduces now that

$$\delta_*: H_i(\mathfrak{M}) \to H_i(\mathfrak{X}) \tag{22}$$

is an isomorphism for $i \neq n + 1 - m$. Statements (c) and (d) follow from the obtained information on (21) and (22).

To prove (e) note that δ_{1*}: $H_{m-1}(\mathfrak{M}) \to H_{m-1}(\mathfrak{X}_1)$ is an epimorphism with kernel $\gamma_*((\partial\xi))$, and α^*: $H^m(\mathfrak{L}) \to H^m(\mathfrak{X})$ is an epimorphism, the kernel of which is an infinite cyclic group generated by some element $\zeta \in H^m(\mathfrak{L})$. Consider the following diagram:

$$\begin{array}{ccc}
H_{m-1}(\mathfrak{X}_1) & & \\
\uparrow \delta_{1*} & & \\
H_{m-1}(\mathfrak{M}) & \underset{\approx}{\xrightarrow{D}} & H^m(\mathfrak{L}) \\
\downarrow \delta_* & & \downarrow \alpha^* \\
H_{m-1}(\mathfrak{X}) & \underset{\approx}{\xrightarrow{D_1}} & H^m(\mathfrak{X})
\end{array}$$

It is commutative up to sign. Let $\theta = D^{-1}(\zeta)$. Then δ_* is an epimorphism with the kernel (θ). The intersection of the subgroups $(\gamma_*(\partial\xi))$ and (θ) is equal to zero since the first of them is finite and the second is isomorphic to Z. So the element $\kappa = \delta_{1*}(\theta)$ has infinite order. Besides,

$$H_{m-1}(\mathfrak{X}_1)/(\kappa) \approx H_{m-1}(\mathfrak{M})/\big(\theta, \gamma_*(\partial\xi)\big)$$

and similarly,

$$H_{m-1}(\mathfrak{X})/(\partial\xi) \approx H_{m-1}(\mathfrak{M})/\big(\theta, \gamma_*(\partial\xi)\big),$$

where $(\theta, \gamma_*(\partial\xi))$ denotes the subgroup generated by θ and $\gamma_*(\partial\xi)$. To base the last relation we have to note once more that $\delta_* \circ \gamma_*(\partial\xi) = \partial\xi$. These two isomorphisms give statement (e).

Proposition 8.15 is proved.

8.18. LEMMA. *The isomorphism constructed in the proof of statement* 8.15(b) *is a P-isomorphism.*

PROOF. Proving statement 8.17(b) we have established that $\delta_{1*} \circ \delta_*^{-1}\colon H_{m-1}(\mathcal{X}) \to H_{m-1}(\mathcal{X}_1)$ is an epimorphism with the kernel $(\partial\xi)$ (we use the notations of subsection 8.4). To prove the lemma it is sufficient to prove that $f = \delta_{1*} \circ \delta_*^{-1}$ is a P-homomorphism. If $a \in H_{m-1}(\mathcal{X})$, then

$$\begin{aligned} f(za) &= \delta_{1*} \circ \delta_*^{-1} \circ \beta_* \circ \alpha_*(a) \qquad \text{(by virtue of (3))} \\ &= \delta_{1*} \circ \delta_*^{-1}(a) - \delta_{1*} \circ \gamma_*(a) \qquad \text{(by virtue of (12))} \\ &= \delta_{1*} \circ \delta_*^{-1}(a) + \beta_{1*} \circ \alpha_*(a) \\ &= \delta_{1*} \circ \delta_*^{-1}(a) + \beta_{1*} \circ \alpha_* \circ \delta_* \circ \delta_*^{-1}(a) \qquad \text{(by virtue of (8))} \\ &= \delta_{1*} \circ \delta_*^{-1}(a) - \beta_{1*} \circ \alpha_{1*} \circ \delta_{1*} \circ \delta_*^{-1}(a) \qquad \text{(by virtue of (10))} \\ &= \delta_{1*} \circ \gamma_{1*} \circ \delta_{1*} \circ \delta_*^{-1}(a) = z_{1*} \circ f(a) = zf(a). \end{aligned}$$

This proves Lemma 8.18.

8.19. PROOF OF THEOREM 8.14. Let $(\mathcal{X}, u, z)$ be any $2q$-isometry with length $\mathcal{X} \leqslant 2$. Then the mapping cones $C(z)$ and $C(\bar{z})$ are $(q-1)$-connected. If the P-module $T_q(\mathcal{X}) = \operatorname{Tors}_{\mathbf{Z}} H_q(\mathcal{X})$ is not minimal, then there exists an element $a \in T_q(\mathcal{X})$, $a \neq 0$, such that $za = 0$ or $\bar{z}a = 0$. Suppose for definiteness that $za = 0$. Then there exists $\eta \in H_{q+1}(C(z))$ with $\Delta_*\eta = a$. By virtue of the Hurewicz Theorem (see 8.3) there exists $\xi \in \sigma_{q+1}(C(z))$ which is taken to η by the Hurewicz homomorphism. Then we have $\partial\xi = a$ (in the notations of 8.11).

Let the $2q$-isometry $(\mathcal{X}_1, u_1, z_1)$ be the modification of the $(\mathcal{X}, u, z)$ corresponding to ξ. According to 8.15(c) the virtual complex $\mathcal{X}_1$ is also $(q-1)$-connected. According to 8.15(e) the order of $T_q(\mathcal{X}_1)$ is less than the order of $T_q(\mathcal{X})$. Producing the just described construction a finite number of times we shall get $2q$-isometry $(\mathcal{X}_s, u_s, z_s)$ such that (a) it is R-equivalent to the initial isometry; (b) length $\mathcal{X}_s \leqslant 2$; (c) the P-module $T_q(\mathcal{X}_s)$ is minimal.

Suppose that the P-module $H_q(\mathcal{X}_s)$ is not minimal. Let us show then there is some primitive element $c \in H_q(\mathcal{X}_s)$ with $zc = 0$ or $\bar{z}c = 0$. If $a \in H_q(\mathcal{X}_s)$, $a \neq 0$ is such that $za = 0$ then a cannot belong to $T_q(\mathcal{X}_s)$. Thus $a = mb$, where $m \in \mathbf{Z}$ and b is a primitive element. Then $m(zb) = 0$ and hence $zb \in T_q(\mathcal{X}_s)$. Since the multiplication by z is a monomorphism $T_q(\mathcal{X}_s) \to T_q(\mathcal{X}_s)$ of the finite group to itself, then it is an epimorphism and so there exists $b_1 \in T_q(\mathcal{X}_s)$ with $zb_1 = zb$. The element $c = b - b_1$ will satisfy all required conditions. Arguments similar to those used in the previous paragraph show that there is a $\zeta \in \sigma_{q+1}(C(z))$ with $\partial\zeta = c$. Let $(\mathcal{X}_{s+1}, u_{s+1}, z_{s+1})$ be the modification of $(\mathcal{X}_s, u_s, z_s)$ corresponding to ζ. Then according to 8.15(a) conn $\mathcal{X}_{s+1} \geqslant q - 1$ and according to 8.15(b) the rank of $H_q(\mathcal{X}_{s+1})$ is less than the rank of $H_q(\mathcal{X}_s)$. The module $T_q(\mathcal{X}_{s+1})$ is also minimal (it follows from the statements 8.15(b) and 8.18). After a finite number of such modifications we obtain an isometry $(\mathcal{X}_r, u_r, z_r)$ with $r > s$ such that (a) it is R-equivalent to the initial isometry: (b) length $\mathcal{X}_r \leqslant 2$; (c) the module $H_q(\mathcal{X}_r)$ is minimal. As it was pointed out in the remark after the formulation of Theorem 8.14, this isometry satisfies all required conditions.

This proves Theorem 8.14.

8.20. PROOF OF THEOREM 7.6. The proof merely follows from the comparision of Propositions 4.4 and 4.5 and Theorem 8.14.

9. The proof of Theorem 7.5. Although Theorem 7.5 is quite similar to Proposition 5.7, we find an essential new difficulty in its proof, caused by the fact that the module B being the part of a P-quintet (A, B, α, l, ψ) with minimal module A can be itself not minimal. That is why we have to consider the modules $(B)_+$ and $(B)_-$ measuring nonminimality of B. These modules $(B)_+$ and $(B)_-$ are extensions of some modules determined by A. The main result of this section which allows us to mange the difficulty mentioned above is Theorem 9.3, stating that the type of these extensions is not invariant of the R-equivalence class.

9.1. Let (A, B, α, l, ψ) be a P-quintet. According to the definition it gives the following exact sequence of P-modules:

$$0 \to A \otimes \mathbf{Z}_2 \xrightarrow{\alpha} B \xrightarrow{\beta} \operatorname{Hom}(A; \mathbf{Z}_2) \to 0.$$

(The P-module structure in $\operatorname{Hom}(A; \mathbf{Z}_2)$ is introduced by the rule: $(zf)(a) = f(\bar{z}a)$, see the *Note* at the end of subsection 7.1.) By virtue of 6.2.IV it generates the exact sequences

$$0 \to (A \otimes \mathbf{Z}_2)_0 \xrightarrow{\alpha_0} (B)_0 \xrightarrow{\beta_0} (\operatorname{Hom}(A; \mathbf{Z}_2))_0 \to 0,$$

$$0 \to (A \otimes \mathbf{Z}_2)_+ \xrightarrow{\alpha_+} (B)_+ \xrightarrow{\beta_+} (\operatorname{Hom}(A; \mathbf{Z}_2))_+ \to 0,$$

$$0 \to (A \otimes \mathbf{Z}_2)_- \xrightarrow{\alpha_-} (B)_- \xrightarrow{\beta_-} (\operatorname{Hom}(A; \mathbf{Z}_2))_- \to 0.$$

The first of these sequences will be called 0 *extension* of the quintet (A, B, α, l, ψ). Similarly, the second will be called $+$ *extension* and the third will be called $-$ *extension* of this quintet. A P-quintet will be called *almost simple* if its $+$ extension and $-$ extension split over $\mathbf{Z}$. A P-quintet will be called *simple* if its $+$ extension and $-$ extension split over P.

Let us note that the form ψ establishes isomorphism between the $+$ extension and the extension consisting of the character groups of the $-$ extension. Thus the $+$ extension splits if and only if the $-$ extension does.

9.2. LEMMA. *Any minimal P-quintet is almost simple.*

PROOF. Let (A, B, α, l, ψ) be a minimal P-quintet. It is sufficient to show that $(B)_+$ and $(B)_-$ are vector spaces over $\mathbf{Z}_2$. According to condition (e) from the definition of P-quintet in 4.1, the multiplication by 2 coincides with the composition

$$B \xrightarrow{\gamma} A \xrightarrow{\pi} A \otimes \mathbf{Z}_2 \xrightarrow{\alpha} B$$

where, for $b \in B$, $\gamma(b)$ satisfies the condition

$$l(\gamma(b) \otimes a) = \psi(b \otimes \alpha(\pi(a)))$$

for all $a \in T(A)$. If $b \in (B)_+$ then $z^n b = 0$ for some n. The minimality of the initial quintet implies minimality of $T(A)$. Therefore $T(A) = (T(A))_0$ and thus for all $a \in T(A)$ there exists an $a_1 \in T(A)$ such that $\bar{z}^n a_1 = a$ (see 9.2.I). So

$$l(\gamma(b) \otimes a) = \psi(b \otimes \alpha(\pi(a))) = \psi(b \otimes \bar{z}^n \alpha(\pi(a_1))) = \psi(z^n b \otimes \alpha(\pi(a_1))) = 0$$

Hence $\gamma(b) = 0$ for $b \in (B)_+$ and thus $2(B)_+ = 0$. Similarly $2(B)_- = 0$ and the lemma is proved.

9.3. THEOREM. *Any P-quintet is R-equivalent to a minimal simple P-quintet.*

PROOF. By virtue of Theorem 7.6 (which was been proved in the preceding section) it is sufficient to show that any minimal P-quintet (A, B, α, l, ψ) is R-equivalent to a minimal simple P-quintet. To do this we construct the sequence of minimal P-quintets in which each quintet is contiguous to the next one and we show that all these quintets beginning with some member are simple.

The first P-quintet $(A_1, B_1, \alpha_1, l_1, \psi_1)$ we define as follows. Put $A_1 = A$ and $l_1(a \otimes b) = l((z\bar{z})^{-1} a \otimes b)$ for a, $b \in T(A_1)$. Since A is minimal then $T(A_1)$ is minimal too and the given definition of l_1 is correct. Define the following P-modules:

$$B^1_+ = \{(b, f) \in (B)_+ \times (\mathrm{Hom}(A; \mathbf{Z}_2))_+ \,;\, \beta(b) = z\bar{z}f\},$$
$$N_1 = \{(\alpha(a), -z\bar{z}a);\, a \in (A \otimes \mathbf{Z}_2)_-\},$$
$$B^1_- = ((B)_- \oplus (A \otimes \mathbf{Z}_2)_-)/N_1.$$

Define the module B_1 as the direct sum

$$B_1 = (B)_0 \oplus B^1_+ \oplus B^1_-.$$

Let us define the homomorphism α_1: $A_1 \otimes \mathbf{Z}_2 \to B_2$ as the direct sum of the following three homomorphisms:

$$(\alpha_1)_0\colon (A_1 \otimes \mathbf{Z}_2)_0 \to (B)_0, \qquad a \mapsto \alpha(a),$$
$$(\alpha_1)_+ \colon (A_1 \otimes \mathbf{Z}_2)_+ \to B^1_+, \qquad a \mapsto (\alpha(a), 0),$$
$$(\alpha_1)_- \colon (A_1 \otimes \mathbf{Z}_2)_- \to B^1_-, \qquad a \mapsto (0, a) + N_1.$$

Let us define the pairing ψ_1: $B_1 \otimes B_1 \to \mathbf{Z}_4$ by the matrix

$$\begin{vmatrix} \kappa & 0 & 0 \\ 0 & 0 & \sigma \\ 0 & \varepsilon\sigma' & 0 \end{vmatrix}$$

where κ: $(B)_0 \otimes (B)_0 \to \mathbf{Z}_4$ is given by the formula $\kappa(a \otimes b) = \psi((z\bar{z})^{-1} a \otimes b)$ for a, $b \in (B)_0$, and σ: $B^1_+ \otimes B^1_- \to \mathbf{Z}_4$ is given by the formula

$$\sigma((b, f) \otimes ((b', c) + N_1)) = \psi(b \otimes b') + f(c)$$

for $(b, f) \in B^1_+$ and $((b', c) + N_1) \in B^1_-$.

So we have a collection $(A_1, B_1, \alpha_1, l_1, \psi_1)$. It is a P-quintet of parity ε. In fact validity of the properties (a), (c), (d), (f), (g) of the P-quintet definition in subsection 4.1 is evident. To verify (b) one has to find the homomorphism β_1: $B_1 \to \mathrm{Hom}(A_1; \mathbf{Z}_2)$ defined by the relation

$$\beta_1(b)(a) = \psi_1(b \otimes \alpha(\pi(a)))$$

for $b \in B_1$, $a \in A_1$. It is easy to see that now β_1 is the direct sum of the following three homomorphisms:

$$(\beta_1)_0\colon (B_1)_0 = (B)_0 \to (\mathrm{Hom}(A_1; \mathbf{Z}_2))_0, \qquad b \mapsto (z\bar{z})^{-1}\beta(b),$$
$$(\beta_1)_+ : B^1_+ \to (\mathrm{Hom}(A_1; \mathbf{Z}_2))_+\,, \qquad (b, f) \mapsto f,$$
$$(\beta_1)_-\colon B^1_- \to (\mathrm{Hom}(A_1; \mathbf{Z}_2))_-, \qquad (\beta_1)_-((b+c) + N_1) = \beta(b).$$

An easy verification shows that there exist exact sequences

$$0 \to (A_1 \otimes \mathbf{Z}_2)_\mu \xrightarrow{(\alpha_1)_\mu} (B_1)_\mu \xrightarrow{(\beta_1)_\mu} (\mathrm{Hom}(A_1; \mathbf{Z}_2))_\mu \to 0$$

where μ means 0, $+$ or $-$. The property (b) now follows from Proposition 6.2.IV.

Let us consider the composition mentioned in the property (e) of the P-quintet definition in subsection 4.1. Its restriction to $(B)_0$ coincides with the similar composition for the initial quintet and thus is the multiplication by 2. The restriction of this composition onto $B^1_+ + B^1_-$ is equal to zero (because of minimality of A_1) and so it coincides with the multiplication by 2 too, since $B^1_+ \oplus B^1_-$ has exponent 2. Thus (3) is satisfied.

To show that the obtained P-quintet $(A_1, B_1, \alpha_1, l_1, \psi_1)$ is contiguous to the initial P-quintet (A, B, α, l, ψ) let us define homomorphisms

$$\varphi\colon A \to A_1, \qquad a \mapsto z\bar{z}a, \quad a \in A,$$
$$\hat{\varphi}\colon A_1 \to A, \qquad a \mapsto a, \quad a \in A_1.$$

Let $\xi\colon B \to B_1$ be the direct sum of the following three homomorphisms:

$$\xi_0\colon (B)_0 \to (B_1)_0, \qquad b \mapsto z\bar{z}b, \quad b \in (B)_0,$$
$$\xi_+ : (B)_+ \to B^1_+\,, \qquad b \mapsto (z\bar{z}b, \beta(b)), \quad b \in (B)_+\,,$$
$$\xi_-\colon (B)_- \to B^1_-, \qquad b \mapsto (b, 0) + N_1, \quad b \in (B)_-.$$

Let $\hat{\xi}\colon B_1 \to B$ be the direct sum of the following three homomorphisms:

$$\hat{\xi}_0\colon (B_1)_0 \to (B)_0, \qquad b \mapsto b, \quad b \in (B_1)_0,$$
$$\hat{\xi}_+ : B^1_+ \to (B)_+\,, \qquad (b, f) \mapsto b, \quad (b, f) \in B^1_+\,,$$
$$\hat{\xi}_-\colon B^1_- \to (B)_-, \qquad (b, a) + N_1 \mapsto \alpha(a) + z\bar{z}b, \quad (b, a) + N_1 \in B^1_-.$$

Then it is easy to verify that φ, $\hat{\varphi}$, ξ, $\hat{\xi}$ satisfy all conditions of the P-quintet continguity definition in subsection 4.1.

So we have just described the construction which, being applied to the minimal P-quintet (A, B, α, l, ψ), gave the contiguous minimal P-quintet $(A_1, B_1, \alpha_1, l_1, \psi_1)$. Let the P-quintet $(A_2, B_2, \alpha_2, l_2, \psi_2)$ be obtained by applying this construction to $(A_1, B_1, \alpha_1, l_1, \psi_1)$. In general, let the P-quintet number i to obtained by applying this construction to the P-quintet number $(i-1)$. By this way we obtain the sequence of P-quintets $(A_i, B_i, \alpha_i, l_i, \psi_i)$, $i = 1, 2, \ldots$. Then all these quintets are minimal and R-equivalent to the intiial one. Let us show that the ith quintet is simple if i is sufficiently large. In fact, B_i can be identified with the direct sum

$$(B)_0 \oplus B^i_+ \oplus B^i_-$$

where

$$B_+^i = \{(b, f) \in (B)_+ \times (\mathrm{Hom}(A;\mathbf{Z}_2))_+ ; \beta(b) = (z\bar{z})^i f\},$$
$$B_-^i = ((B)_- \oplus (A \otimes \mathbf{Z}_2)_-)/N_i,$$
$$N_i = \{(\alpha(a), -(z\bar{z})^i a); a \in (A \otimes \mathbf{Z}_2)_-\}.$$

This implies that if i is so large that $(z\bar{z})^i(A \otimes \mathbf{Z}_2)_- = 0$ then B_+^i is isomorphic to the direct sum $(A \otimes \mathbf{Z}_2)_+ \oplus (\mathrm{Hom}(A;\mathbf{Z}_2))_+$ and besides, $(\alpha_i)_+$ is the inclusion of the first summand and $(\beta_i)_+$ is the projection on the second summand. For such i, N_i consists of all pairs of the form $(\alpha(a), 0)$ where $a \in (A \otimes \mathbf{Z}_2)_+$ and B_-^i is isomorphic to $(\mathrm{Hom}(A;\mathbf{Z}_2))_- \oplus (A \otimes \mathbf{Z}_2)_-$ and besides, $(\alpha_i)_-$ is the inclusion of the second summand and $(\beta_i)_-$ is the projection on the first summand. But this means that the ith quintet $(A_i, B_i, \alpha_i, l_i, \psi_i)$ is simple.

Theorem 9.3 is proved.

9.4. LEMMA. *Any two minimal simple P-quintets* $(A_\nu, B_\nu, \alpha_\nu, l_\nu, \psi_\nu)$, $\nu = 1, 2$, *are contiguous if and only if there are P-homorphisms*

$$\varphi: A_1 \to A_2, \quad \hat{\varphi}: A_2 \to A_1, \quad \xi_0: (B_1)_0 \to (B_2)_0, \quad \hat{\xi}_0: (B_2)_0 \to (B_1)_0$$

such that the following conditions hold:

(a) *the two diagrams below are commutative*:

$$\begin{array}{ccc} (A_1 \otimes \mathbf{Z}_2)_0 & \xrightarrow{(\varphi \otimes 1)_0} & (A_2 \otimes \mathbf{Z}_2)_0 \\ (\alpha_1)_0 \downarrow & & \downarrow (\alpha_2)_0 \\ (B_1)_0 & \xrightarrow{\xi_0} & (B_2)_0 \end{array} \qquad \begin{array}{ccc} (A_2 \otimes \mathbf{Z}_2)_0 & \xrightarrow{(\hat{\varphi} \otimes 1)_0} & (A_1 \otimes \mathbf{Z}_2)_0 \\ (\alpha_2)_0 \downarrow & & \downarrow (\alpha_1)_0 \\ (B_2)_0 & \xrightarrow{\hat{\xi}_0} & (B_1)_0 \end{array}$$

(b) $\psi_1 \circ (\hat{\xi}_0 \otimes 1_{(B_1)_0}) = \psi_2 \circ (1_{(B_2)_0} \otimes \xi_0)$,

(c) $l_1 \circ (\hat{\varphi}|_{T(A)} \otimes 1_{T(A_1)}) = l_2 \circ (1_{T(A_2)} \otimes \varphi|_{T(A_1)})$,

(d) *the homomorphisms* $\varphi \circ \hat{\varphi}$, $\hat{\varphi} \circ \varphi$, $\xi_0 \circ \hat{\xi}_0$, $\hat{\xi}_0 \circ \hat{\xi}_0$, $\hat{\xi}_0 \circ \xi_0$ *all coincide with multiplication by* $z\bar{z} \in P$.

PROOF. Contiguity clearly implies the formulated conditions. To prove the inverse statement let us identify B_ν, $\nu = 1, 2$, with the direct sum

$$(B_\nu)_0 \oplus (A_\nu \otimes \mathbf{Z}_2)_+ \oplus (A_\nu \otimes \mathbf{Z}_2)_- \oplus (\mathrm{Hom}(A_\nu;\mathbf{Z}_2))_+ \oplus (\mathrm{Hom}(A_\nu;\mathbf{Z}_2))_-.$$

If this decomposition is obtained by splitting the $+$ extension and $-$ extension, then ψ_ν will be given by the matrix

$$\begin{vmatrix} (\psi_\nu)_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \varepsilon\delta \\ 0 & 0 & 0 & \varepsilon\sigma & 0 \\ 0 & 0 & \sigma' & 0 & 0 \\ 0 & \delta' & 0 & 0 & 0 \end{vmatrix}$$

where $(\psi_\nu)_0: (B_\nu)_0 \otimes (B_\nu)_0 \to \mathbf{Z}_4$ is the restriction of ψ_ν, and σ and δ are restrictions of the canonical evaluation pairing

$$(A_\nu \otimes \mathbf{Z}_2) \otimes \mathrm{Hom}(A_\nu;\mathbf{Z}_2) \to \mathbf{Z}_2 \overset{\subset}{\to} \mathbf{Z}_4.$$

Define homomorphisms $\xi: B_1 \to B_2$, $\hat{\xi}: B_2 \to B_1$ as the direct sums

$$\xi = \xi_0 \oplus (\varphi \otimes 1)_+ \oplus (\varphi \otimes 1)_- \oplus (\hat{\varphi}^*)_+ \oplus (\hat{\varphi}^*)_-,$$
$$\hat{\xi} = \hat{\xi}_0 \oplus (\hat{\varphi} \otimes 1)_+ \oplus (\hat{\varphi} \otimes 1)_- \oplus (\varphi^*)_+ \oplus (\varphi^*)_-.$$

Here the star means the dual homomorphisms in the character theory sense. Then for φ, $\hat{\varphi}$, ξ, $\hat{\xi}$ all conditions of the definition of P-quintet contiguity (see subsection 4.1) are satisfied and so the lemma follows.

9.5. Proof of Theorem 7.5. One part of the theorem is trivial. In fact, if for all i the P-quintets $(A_i, B_i, \alpha_i, l_i, \psi_i)$ and $(A_{i+1}, B_{i+1}, \alpha_{i+1}, l_{i+1}, \psi_{i+1})$ are contiguous, where $i = 1, 2, \dots, n-1$, then let

$$\varphi_i: A_i \to A_{i+1}, \qquad \xi_i: B_i \to B_{i+1}, \qquad \hat{\varphi}_i: A_{i+1} \to A_i, \qquad \hat{\xi}_i: B_{i+1} \to B_i$$

be the homomorphisms realizing contiguity of these quintets. Then one can define the homomorphisms

$$\varphi: A_1 \to A_n, \qquad \hat{\varphi}: A_n \to A_1, \qquad \xi: B_1 \to B_n, \qquad \hat{\xi}: B_n \to B_1$$

as the following compositions:

$$\varphi = \varphi_{n-1} \circ \cdots \circ \varphi_2 \circ \varphi_1, \qquad \xi = \xi_{n-1} \circ \cdots \circ \xi_2 \circ \xi_1,$$
$$\hat{\varphi} = \hat{\varphi}_1 \circ \cdots \circ \hat{\varphi}_{n-2} \circ \hat{\varphi}_{n-1}, \qquad \hat{\xi} = \hat{\xi}_1 \circ \cdots \circ \hat{\xi}_{n-2} \circ \hat{\xi}_{n-1}.$$

It is easy to see that φ, ξ, $\hat{\varphi}$, $\hat{\xi}$ satisfy all conditions of Theorem 7.5 (for the first and for the nth of these quintets).

Conversely, suppose we are given two P-quintets $(A_\nu, B_\nu, \alpha_\nu, l_\nu, \psi_\nu)$, $\nu = 1, 2$, and homomorphisms

$$\varphi: A_1 \to A_2, \qquad \xi: B_1 \to B_2, \qquad \hat{\varphi}: A_2 \to A_1, \qquad \hat{\xi}: B_2 \to B_1$$

satisfying all conditions of Theorem 7.5. We have to prove that these quintets are R-equivalent. By virtue of Theorem 9.3 we may suppose that these quintets are minimal and simple. The proof of the theorem will be obtained by the induction on m, where m is the number from conditions (d) of Theorem 7.5. If $m = 1$, then the condition of Theorem 7.5 coincide with the definition of P-quintet contiguity (see 4.1), and so in this case the theorem is true. Suppose we have already proved this theorem for all m less than some integer n and let us consider the case with $m = n$.

We shall construct a new P-quintet $(A^1, B^1, \alpha^1, l^1, \psi^1)$ such that

(1) it is minimal and simple;

(2) it is contiguous to $(A_1, B_1, \alpha_1, l_1, \psi_1)$;

(3) there exist homomorphisms

$$\varphi^1: A^1 \to A_2, \qquad \xi^1: B^1 \to B_2, \qquad \hat{\varphi}^1: A_2 \to A^1, \qquad \hat{\xi}^1: B_2 \to B^1$$

satisfying conditions (a), (b), (c) of Theorem 7.5 and condition (d) with $m = n - 1$. Then by the inductive hypothesis, $(A^1, B^1, \alpha^1, l^1, \psi^1)$ is R-equivalent to $(A_2, B_2, \alpha_2, l_2, \psi_2)$. Thus the initial P-quintets are R-equivalent and so from the existence of $(A^1, B^1, \alpha^1, l^1, \psi^1)$ Theorem 7.5 follows.

Let A^1 be the set of all $a \in A_2$ such that $z\bar{z}a \in \operatorname{im} \varphi$. Let incl: $A^1 \to A_2$ denote the natural inclusion. The minimality of A_2 implies $T(A^1) = T(A_2)$. Thus we can define the form $l^1\colon T(A^1)\otimes T(A^1)\to \mathbf{Q}/\mathbf{Z}$ by $l^1(x\otimes y)=l_2(x\otimes(z\bar z)^{-n+1}y)$, where $x, y \in T(A^1)$. Define B^1 as the direct sum decomposition

$$B^1=(B_2)_0\oplus(A^1\otimes\mathbf{Z}_2)_+\oplus(A^1\otimes\mathbf{Z}_2)_-\oplus(\operatorname{Hom}(A^1;\mathbf{Z}_2))_+\oplus(\operatorname{Hom}(A^1;\mathbf{Z}_2))_-.$$

Define $\alpha^1\colon A^1\otimes\mathbf{Z}_2\to B^1$ as the direct sum of the following homomorphisms:

$$\alpha_0^1\colon (A^1\otimes\mathbf{Z}_2)_0\xrightarrow{(\mathrm{incl}\otimes 1)_0}(A_2\otimes\mathbf{Z}_2)_0\xrightarrow{(\alpha_2)_0}(B_2)_0,$$

$$\alpha_+^1\colon (A^1\otimes\mathbf{Z}_2)_+\to B^1,\qquad \alpha_-^1\colon (A^1\otimes\mathbf{Z}_2)_-\to B^1$$

where α_+^1 and α_-^1 are inclusions on the corresponding summands. Define the form $\psi^1\colon B^1\otimes B^1\to\mathbf{Z}_4$ by the matrix

$$\begin{vmatrix} \kappa & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \varepsilon\sigma\\ 0 & 0 & 0 & \varepsilon\delta & 0\\ 0 & 0 & \delta' & 0 & 0\\ 0 & \sigma' & 0 & 0 & 0 \end{vmatrix}$$

where $\kappa(x\otimes y)=\psi_2(x\otimes(z\bar z)^{-n+1}y)$, $x, y\in(B_2)_0$ and σ and δ are the restrictions of the canonical evaluation pairing

$$(A^1\otimes\mathbf{Z}_2)\otimes\operatorname{Hom}(A^1;\mathbf{Z}_2)\to\mathbf{Z}_2\xrightarrow{\subset}\mathbf{Z}_4.$$

The prime denotes transposition as usual.

Let us note that the homomorphism $(\mathrm{incl}\otimes 1)_0\colon (A^1\otimes\mathbf{Z}_2)_0\to(A_2\otimes\mathbf{Z}_2)_0$, participating in the definition of α_0^1, is an isomorphism. In fact, if $a\in A_2$, then $(z\bar z)^n a=\varphi(\hat\varphi(a))$ and so $(z\bar z)^{n-1}a\in A^1$. Define $f\colon A_2\to A^1$ by $f(a)=(z\bar z)^{n-1}a$, for $a\in A_2$. Then the homomorphism $(z\bar z)^{1-n}(f\otimes 1)_0$ is clearly inverse to $(\mathrm{incl}\otimes 1)_0$.

We have constructed the collection $(A^1, B^1, \alpha^1, l^1, \beta^1)$. Let us prove that it is a P-quintet of parity ε. Conditions (a), (c), (d), (f), (g) of the definition of P-quintet (see subsection 4.1) are obviously satisfied and we only have to verify (b) and (e). To do this let us note that the homomorphism $\beta^1\colon B^1\to\operatorname{Hom}(A^1;\mathbf{Z}_2)$ defined by the formula $\beta^1(b)(a)=\psi^1(b\otimes\alpha^1(\pi(a)))$ for $a\in A^1$, $b\in B^1$, is the direct sum of the following three homomorphisms:

$$(\beta^1)_0\colon (B_2)_0\xrightarrow{(\beta_2)_0}(\operatorname{Hom}(A_2;\mathbf{Z}_2))_0\underset{\approx}{\xrightarrow{(\mathrm{incl}^*)_0}}(\operatorname{Hom}(A^1;\mathbf{Z}_2))_0$$

$$\underset{\approx}{\xrightarrow{(z\bar z)^{-n+1}}}(\operatorname{Hom}(A^1;\mathbf{Z}_2))_0,$$

$$(\beta^1)_+\colon (B^1)_+\to(\operatorname{Hom}(A^1;\mathbf{Z}_2))_+,\qquad (\beta^1)_-\colon (B^1)_-\to(\operatorname{Hom}(A^1;\mathbf{Z}_2))_-,$$

where $(\beta^1)_+$ and $(\beta^1)_-$ are the projections on the corresponding summands. As in the previous paragraph, one can show that $(\mathrm{incl}^*)_0$ is an isomorphism. Thus the sequence

$$0\to(A^1\otimes\mathbf{Z}_2)_0\xrightarrow{(\alpha^1)_0}(B^1)_0\xrightarrow{(\beta^1)_0}(\operatorname{Hom}(A^1;\mathbf{Z}_2))_0\to 0$$

is isomorphic to the 0 extension of the quintet $(A_2, B_2, \alpha_2, l_2, \psi_2)$ and so it is exact.

The similar $+$ and $-$ sequences are also exact (they even split over P). Now the property (b) follows from Propostion 6.2.IV.

To verify (e) note that the restriction of $\gamma^1\colon B^1 \to T(A^1)$ to $(B^1)_+ \oplus (B^1)_-$ is equal to zero (this can be deduced from the minimality of A_2 exactly by the same way as it was done in the proof of Lemma 9.2). The restriction of γ^1 on $(B^1)_0$ coincides with $(\gamma_2)_0\colon (B_2)_0 \to T(A_2) = T(A^1)$. From this it follows that the composition

$$B^1 \xrightarrow{\gamma^1} T(A^1) \xrightarrow{\pi} A^1 \otimes \mathbf{Z}_2 \xrightarrow{\alpha^1} B^1$$

coincides with multiplication by 2. In fact, it is equal to zero on $(B^1)_+ + (B^1)_-$ but this group has exponent 2, and on $(B^1)_0$ it is equal to the similar composition for $(A_2, B_2, \alpha_2, l_2, \psi_2)$ which is multiplication by 2 by hypothesis.

Thus the collection $(A^1, B^1, \gamma^1, l^1, \psi^1)$ is a P-quintet. It is, of course, simple. Since A^1 is a submodule of A_2, it is also minimal. To show that it is contiguous to $(A_1, B_1, \alpha_1, l_1, \psi_1)$ define $\varphi_1\colon A_1 \to A^1$ and $\hat{\varphi}_1\colon A^1 \to A_1$ as follows: φ_1 is equal to φ and $\hat{\varphi}_1(x) = \varphi^{-1}(z\bar{z}x)$ for $x \in A^1$. Let us define $\xi_0\colon (B_1)_0 \to (B^1)_0$ as $\xi|_{(B_1)_0}$. Then ξ_0 is an isomorphism. Let $\hat{\xi}_0\colon (B^1)_0 \to (B_1)_0$ be given by the formula $\hat{\xi}_0(x) = \xi_0^{-1}(z\bar{z}x)$ for $x \in (B^1)_0$. Then all conditions of Lemma 9.4 are satisfied and by virtue of this lemma $(A_1, B_1, \alpha_1, l_1, \psi_1)$ is contiguous to $(A^1, B^1, \alpha^1, l^1, \psi^1)$.

Define homomorphisms $\varphi^1\colon A^1 \to A_2$, $\hat{\varphi}^1\colon A_2 \to A^1$ by the formulas $\varphi^1(a) = a$ for $a \in A^1$ and $\hat{\varphi}^1(b) = (z\bar{z})^{n-1}b$ for $b \in A_2$. Let $\xi_0^1\colon (B^1)_0 \to (B_2)_0$ be the identity map and $\hat{\xi}_0^1\colon (B_2)_0 \to (B^1)_0$ be the multiplication by $(z\bar{z})^{n-1} \in P$. Using direct sum decompositions, quite similarly to the method of the proof of Lemma 9.4, one can construct homomorphisms $\xi^1\colon B^1 \to B_2$ and $\hat{\xi}^1\colon B_2$ and $\hat{\xi}^1\colon B_2 \to B^1$ extending ξ_0^1 and $\hat{\xi}_0^1$ correspondingly and such that φ^1, ξ^1, $\hat{\varphi}^1$, $\hat{\xi}^1$ satisfy Theorem 7.5 and besides, in (d), $m = n - 1$. Thus the step of induction is done.

Theorem 7.5 is proved.

10. Generalized homology of knot complement's infinite cyclic covering. In this section there will be established the formula

$$h_j(\hat{X}) \approx h_j(V) \otimes_P L$$

expressing (generalized) homology module (over ring $\Lambda = \mathbf{Z}[t, t^{-1}]$) of knot complement's infinite cyclic covering $\hat{X}$ in terms of the homology module (over ring $P = \mathbf{Z}[z]$) of any Seifert manifold V of this knot. The P-module structure on $h_j(V)$ used here is induced by the S-map $z\colon V \to V$ which is the part of the n-isometry of the n-isometry of V (defined in §2). In the case of middle-dimensional singular homology group $H_q(V)$ the same module structure was introduced by Kervaire **[8]**, see also Stoltzfus **[16]**. This formula will be applied in the next section to obtain a geometrical interpretation of the modules from the L-quintet of a knot.

As a conclusion we outline in subsection 10.8 the solution of a more general problem of constructing the equivariant stable homotopy type of $\hat{X}$ by means of the n-isometry (V, u, z) of any Seifert manifold V of this knot. To give the precise formulation of this result (which will not be used in the paper) we would have to describe an equivariant analogy of the category Stab_0. This will be done elsewhere.

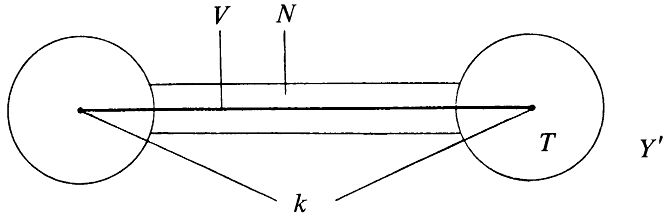

FIGURE 1

10.1. Let (S^{n+2}, k^n) be any spherical knot and let $V^{n+1} \subset S^{n+2}$ be an arbitrary Seifert manifold of this knot. Consider the tubular neighbourhood T of k^n in S^{n+2} which we suppose to be so small that $T \cap V$ is a collar of the boundary ∂V. Let $X = S^{n+2} - \operatorname{int}(T)$ and $p\colon \tilde{X} \to X$ be the infinite cyclic covering. Consider some meridian m on ∂X (i.e. a simple closed curve which is the boundary of the fibre of the normal bundle of k^n). We shall also suppose that m meets V transversally and $V \cap m = v_0$, where v_0 is the base point of V. The preimage $\tilde{m} = p^{-1}$ is homeomorphic to R. Orient m so that it would have the linking coefficient $+1$ with k. The orientation of m generates an orientation of $\tilde{m}$. Let $t\colon \tilde{X} \to \tilde{X}$ be that of the two generators of the covering transformation group of p for which $t|_{\tilde{m}}\colon \tilde{m} \to \tilde{m}$ is a displacement in the positive direction. Denote $\hat{X} = \tilde{X}/\tilde{m}$. This space will be called *reduced infinite cycle covering.*

If h_* is any generalized homology theory on Stab_0 then we define $h_i(\hat{X})$ as the direct limit of the groups $h_i(K/K \cap \tilde{m})$, where K runs over all finite subcomplexes $K \subset \tilde{X}$. The covering transformation t is a cellular map of $\hat{X}$ to itself and it preserves the base point. So it induces an automorphism of $h_i(\hat{X})$ and thus $h_i(\hat{X})$ is a module over the ring $\Lambda = \mathbf{Z}[t, t^{-1}]$.

It is well known that for singular homology theory the homomorphisms

$$t - 1\colon H_i(\hat{X}) \to H_i(\hat{X})$$

are all isomorphisms. Applying the suspension, we can construct a continuous map $St - S1\colon S\hat{X} \to S\hat{X}$. This map induces isomorphisms of all singular homology groups and thus it is a homotopy equivalence. This implies that for any homology theory h_* all homomorphisms

$$t - 1\colon h_i(\hat{X}) \to h_i(\hat{X})$$

are isomorphisms and so Λ-modules $h_i(\hat{X})$ can be considered as L-modules, where $L = \mathbf{Z}[t, t^{-1}, (1-t)^{-1}] = \mathbf{Z}[z, z^{-1}, \bar{z}^{-1}]$. (Recall that $z = (1-t)^{-1}$.)

Let (V, u, z) be the n-isometry of the Seifert manifold V (see 2.5). Then the S-map $z\colon V \to V$ induces an endomorphism of $h_i(V)$, and so $h_i(V)$ is a $P = \mathbf{Z}[z]$-module. Our aim now is to find a relation between the P-module $h_i(V)$ and the L-module $h_i(\hat{X})$. Let $W = V \cap X$ and $i\colon W \to \hat{X}$ be the composition of any lifting of the inclusion $W \to X$ and the canonical projection $\tilde{X} \to \hat{X}$. Let $r\colon V \to W$ be a retraction contracting the collar $V - \operatorname{int}(W)$ to ∂W.

10.2. THEOREM. (1) *The homomorphism* $\varphi\colon h_j(V) \to h_j(\hat{X})$, *where* $\varphi(a) = i_* r_*(a)$ *for* $a \in h_j(V)$, *is a P-homomorphism.*

(2) *The L-module homomorphism*

$$\Phi\colon h_j(V) \otimes_P L \to h_j(\hat{X}),$$

where $\Phi(a \otimes 1) = \varphi(a)$ *for* $a \in h_j(V)$, *is an isomorphism.*

10.3. PROOF OF STATEMENT (1) OF THEOREM 10.2. Let N be a small tubular neighbourhood of W in X and denote $Y' = S^{n+2} - \operatorname{int}(N)$ (see Figure 1). Define $Y = Y'/Y' \cap m$. Let $v\colon W \wedge Y \to S^{n+1}$ be the canonical Spanier-Whitehead duality map and $i_+\colon W \to Y$, $i_-\colon W \to Y$ be small translations in the directions of positive and negative normals to W correspondingly. Then, according to [**2**, §1], the Seifert homotopy pairing of W is given by $\theta = v \circ (1 \otimes i_+)$. As it was shown in [**2**, p. 188], θ' is homotopic to $(-1)^n v \circ (1 \otimes i_-)$. If (W, u, z) is the n-isometry corresponding to the spherical pairing (W, θ) (see §2), then $u = \theta + (-1)^{n+1}\theta' = v \circ (1 \otimes i_+) - v \circ (1 \otimes i_-) = v \circ (1 \otimes (i_+ - i_-))$. Since u is a duality map, $i_+ - i_-\colon W \to Y$ is an equivalence in Stab_0.

As it was pointed out in 2.4, the S-map $z\colon W \to W$ satisfies $\theta = u \circ (1 \otimes z)$. This is equivalent to $v \circ (1 \otimes (i_+ - i_-) \circ z) = \theta = v \circ (1 \otimes i_+)$. Since v is a duality map,

$$(i_+ - i_-) \circ z = i_+ . \tag{1}$$

This implies $(i_+ - i_-) \circ \bar{z} = i_+ - i_- - (i_+ - i_-) \circ z = -i_-$, i.e.

$$(i_+ - i_-) \circ \bar{z} = -i_- . \tag{2}$$

Let us consider the map $l\colon Y \to \hat{X}$ which is the composition of any lifting of the inclusion $Y' \to X$ and the projection $\tilde{X} \to \hat{X}$. This lifting may be chosen such that $l \circ i_+$ would be homotopic to i. Then clearly $l \circ i_-$ is homotopic to $t \circ l \circ i_+$. Applying l to the left and right sides of (1) we obtain the following equality for the stable homotopy classes:

$$(1 - t) \circ i \circ z = i. \tag{3}$$

If h_* is any homology theory and $a \in h_j(W)$ then (3) implies $(1 - t)i_*(za) = i_*(a)$ or $i_*(za) = (1 - t)^{-1} i_*(a) = z i_*(a)$. This proves statement (1) of Theorem 10.2.

10.4. In this subsection we construct one Puppe sequence which will be helpful in the proof of the second statement of Theorem 10.2. Here we shall use the constructions made in subsections 10.1 and 10.3 and the notations introduced there.

Let $\mathbf{Z}^+$ denote the space which is the union of the set of integers Z (with the discrete topology) and of the base point $*$. The points of $Y \wedge \mathbf{Z}^+$ can be presented as pairs (y, n), where $y \in Y$, $n \in \mathbf{Z}$ with identification $(y_0, n) = (y_0, m)$ for all $n, m \in \mathbf{Z}$ (here $y_0 = \operatorname{im} \cap Y'/m \cap Y'$ is the base point of Y). The infinite cyclic group with generator t acts on the space $Y \wedge \mathbf{Z}^+$, where $t(y, n) = (y, n + 1)$.

Consider the map

$$J\colon Y \wedge \mathbf{Z}^+ \to \hat{X},$$

where $J(y, n) = t^n \circ l(y)$ and $l\colon Y \to \hat{X}$ is the map introduced in the previous subsection. If $y = y_0$ is the base point of Y then $l(y_0) = x_0$ is the base point of $\hat{X}$, and so this formula correctly defines J since $t(x_0) = x_0$. J is clearly an equivariant map.

Let us consider the Puppe sequence of J,

$$Y \wedge \mathbf{Z}^+ \xrightarrow{J} \hat{X} \xrightarrow{E} C(J) \xrightarrow{D} C(E) \to S\hat{X}.$$

Here $C(J) = C(Y \wedge \mathbf{Z}^+) \cup_J \hat{X}$ is the mapping cone of J. The points of the cone $C(Y \wedge \mathbf{Z}^+)$ can be written in the form of $\tau \wedge y \wedge n$, where $\tau \in [0, 1]$, $y \in Y$, $n \in \mathbf{Z}$, and besides, $0 \wedge y_1 \wedge n_1 = 0 \wedge y_2 \wedge n_2$, $\tau_1 \wedge y_0 \wedge n_1 = \tau_2 \wedge y_0 \wedge n_2$ for all $y_1, y_2 \in Y$, $n_1, n_2 \in \mathbf{Z}$. The points $1 \wedge n \wedge y$ must be identified with $t^n \circ l(y) \in \hat{X}$. The map E is the natural inclusion. $C(E)$ and D are defined similarly.

Let κ: $N \to \hat{X}$ be the composition of that lifting of the inclusion $N \subset X$ and the projection $\tilde{X} \to \hat{X}$ such that $\kappa \circ i_+ = l \circ i_+$. Then $\kappa \circ i_- = t^{-1} \circ l \circ i_-$. Let ρ: $W \times [-1, 1] \to N$ be a homeomorphism such that $\rho(w, 0) = w$, $\rho(w, 1) = i_+(w)$, $\rho(w, -1) = i_-(w)$ for all $w \in W$. Then we can define map α: $SW \wedge \mathbf{Z}^+ \to C(J)$ by the formula

$$\alpha(\tau \wedge w \wedge n) = \begin{cases} (3\tau) \wedge i_+(w) \wedge n & \text{for } 0 \leqslant \tau \leqslant 1/3, \\ t^n \circ \kappa(\rho(w, 6(1/2 - \tau))) & \text{for } 1/3 \leqslant \tau \leqslant 2/3, \\ (3(1-\tau)) \wedge i_-(w) \wedge (n-1) & \text{for } 2/3 \leqslant \tau \leqslant 1. \end{cases}$$

Symbol SW means here the reduces suspension over W, i.e. $SW = ([0, 1]/\{0, 1\}) \wedge W$. $C(J)$ is schematically shown in Figure 2, and $SW \wedge \mathbf{Z}^+$ is shown in Figure 3. Geometrically α may be described as follows: it is simply an inclusion of the second picture into the first one. It is clear that α is an equivariant homotopy equivalence.

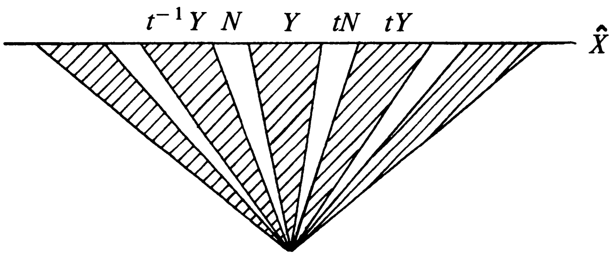

FIGURE 2. $C(J)$

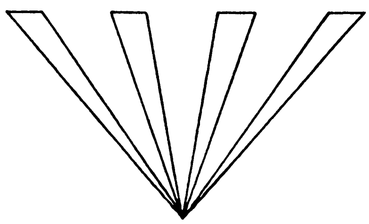

FIGURE 3. $SW \wedge \mathbf{Z}^+$

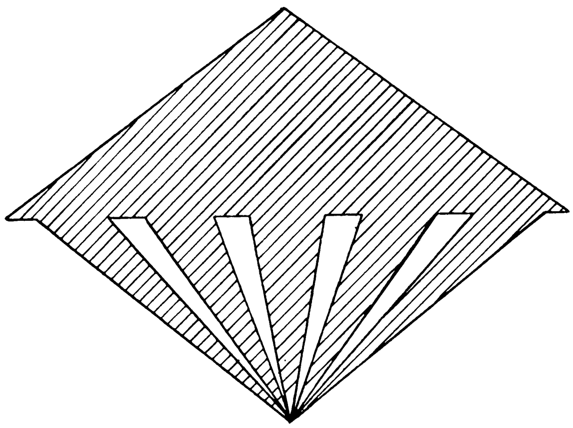

FIGURE 4. $C(E)$

The complex $C(E)$ is the union $C(J) \cup C(\hat{X})$, where every point of $\hat{X} \subset C(\hat{X})$ is identified with the same point in $C(J)$ (see Figure 4). It is clear that $C(E)/C(\hat{X})$ is homeomorphic to $SY \wedge \mathbf{Z}^+$. Let β: $C(E) \to SY \wedge \mathbf{Z}^+$ be the corresponding projection. Then β is an equivariant homotopy equivalence too.

Let us consider the composition $\beta \circ D \circ \alpha$. It is given by the formula

$$\beta \circ D \circ \alpha(\tau \wedge w \wedge n) = \begin{cases} (3\tau) \wedge i_+(w) \wedge n & \text{for } 0 \leqslant \tau \leqslant 1/3, \\ * & \text{for } 1/3 \leqslant \tau \leqslant 2/3, \\ (3(1-\tau)) \wedge i_-(w) \wedge (n-1) & \text{for } 2/3 \leqslant \tau \leqslant 1. \end{cases}$$

This map is equivariantly homotopic to the map Δ: $SW \wedge \mathbf{Z}^+ \to SY \wedge \mathbf{Z}^+$ given by the formula

$$(4) \qquad \Delta(\tau \wedge w \wedge n) = \begin{cases} (2\tau) \wedge i_+(w) \wedge n & \text{for } 0 \leqslant \tau \leqslant 1/2, \\ 2(1-\tau) \wedge i_-(w) \wedge (n-1) & \text{for } 1/2 \leqslant \tau \leqslant 1. \end{cases}$$

Since α and β are equivariant homotopy equivalences, the initial Puppe sequence gives us the following sequence, consisting of equivariant (relative to the action of $\mathbf{Z}$) complexes and maps:

$$(5) \qquad Y \wedge \mathbf{Z}^+ \xrightarrow{\bar{\partial}} \hat{X} \to SW \wedge \mathbf{Z}^+ \xrightarrow{\Delta} SY \wedge \mathbf{Z}^+ \to S\hat{X}.$$

It is exact in the sense explained in subsection 8.2, i.e. it induces exact sequence of abelian groups in any homology and cohomology theory.

10.5. Let h_* be a homology theory on the category Stab_0. Taking the inductive limit over all finite subcomplexes it may be extended to all CW-complexes with base points. It is clear that for any space $\hat{X}$ with a fixed $\mathbf{Z}$-action preserving the base point, the group $h_i(\hat{X})$ has a natural Λ-module structure.

For the $\mathbf{Z}$-spaces $Y \wedge \mathbf{Z}^+$ and $SW \wedge \mathbf{Z}^+$ we have the following Λ-isomorphisms:

$$h_j(Y \wedge \mathbf{Z}^+) \approx h_j(Y) \otimes_{\mathbf{Z}} \Lambda, \qquad h_j(SW \wedge \mathbf{Z}^+) \approx h_{j-1}(W) \otimes_{\mathbf{Z}} \Lambda.$$

Now applying h_{j+1} to the exact sequence (5) we obtain the following piece of exact sequence of modules and homomorphisms over Λ:

$$h_j(W) \otimes_{\mathbf{Z}} \Lambda \xrightarrow{d} h_j(Y) \otimes_{\mathbf{Z}} \Lambda \to h_j(\hat{X})$$

where d is induced by Δ. The formula (4) implies that, for $a \in h_j(W)$, $d(a \otimes 1) = i_{+*}(a) \otimes t - t_{-*}(a) \otimes 1$.

10.6. LEMMA. *The homomorphism d is a monomoirphism.*

PROOF. If $a \in h_j(W)$ then $d(a \otimes 1) = i_{+*}(a) \otimes (t-1) + (i_+ - i_-)_*(a) \otimes 1$. If $\ker(d) \neq 0$ then $\ker(d)$ contains an element p of the from

$$p = \sum_{k=n}^{m} a_k \otimes (t-1)^k$$

where $m \geqslant n \geqslant 0$, $a_k \in h_j(W)$, $a_n \neq 0$. Then

$$0 = d(p) = \sum_{k=n}^{m} i_{+*}(a_k) \otimes (t-1)^{k+1} + \sum_{k=n}^{m} (i_+ - i_-)_*(a_k) \otimes (t-1)^k.$$

From this it follows that $(i_+ - i_-)_*(a_n) = 0$. Since $i_+ - i_-$ is a stable homotopy equivalence (see 10.3) it implies $a_n = 0$, which is a contradiction.

10.7. PROOF OF STATEMENT (2) OF THEOREM 10.2. By virtue of the results of subsection 10.5 and Lemma 10.6 there exists the exact sequence

$$0 \to h_j(W) \otimes_{\mathbf{Z}} \lambda \xrightarrow{d} h_j(Y) \otimes_{\mathbf{Z}} \Lambda \xrightarrow{e} h_j(\hat{X}) \to 0.$$

Here let us substitute $h_j(Y)$ for $h_j(W)$, identifying these groups by means of the isomorphism $(i_+ - i_-)_*\colon h_j(W) \to h_j(Y)$. We obtain the exact sequence

$$0 \to h_j(W) \otimes_{\mathbf{Z}} \Lambda \xrightarrow{d'} h_j(W) \otimes_{\mathbf{Z}} \Lambda \to h_j(\hat{X}) \to 0.$$

Using (1) and (2) from 10.3 it is easy to see that d' acts as follows: for $a \in h_j(W)$, $d'(a \otimes 1) = za \otimes t + \bar{z}a \otimes 1$. Multiplying this exact sequence by $\otimes_\Lambda L$ and using the fact that $h_j(\hat{X})$ is an L-module we get the exact sequence

$$h_j(W) \otimes_{\mathbf{Z}} L \xrightarrow{d''} h_j(W) \otimes_{\mathbf{Z}} L \to h_j(\hat{X}) \to 0 \tag{6}$$

where d'' acts by the same formula as d' does. On the other hand there is the exact sequence of P-modules

$$h_j(W) \otimes_{\mathbf{Z}} P \xrightarrow{\nabla} h_j(W) \otimes_{\mathbf{Z}} P \xrightarrow{H} h_j(W) \to 0 \tag{7}$$

where $\nabla(a \otimes 1) = a \otimes z - za \otimes 1$, $H(a \otimes 1) = a$ for $a \in h_j(W)$. Multiplying (7) by $\otimes_P L$ we obtain the exact sequence

$$h_j(W) \otimes_{\mathbf{Z}} L \xrightarrow{\nabla'} h_j(W) \otimes_{\mathbf{Z}} L \xrightarrow{H'} h_j(W) \otimes_P L \to 0 \tag{8}$$

where, for $a \in h_j(W)$,

$$\begin{aligned}\nabla'(a \otimes 1) &= a \otimes z - za \otimes 1 = z(a \otimes 1 - za \otimes z^{-1}) \\ &= z(a \otimes 1 - za \otimes (1-t)) = z(za \otimes t - (a - za) \otimes 1) = zd''(a).\end{aligned}$$

Therefore $\nabla' = zd''$ and comparing (6) and (8) we get the L-isomorphism

$$h_j(W) \otimes_P L \underset{\approx}{\rightarrow} h_j(\hat{X}).$$

Composing it with $r_* \otimes 1$ we obtain Φ. Thus Theorem 10.2 is proved.

10.8. Here we briefly outline a further result. It will not be used in the paper and here we do not present its precise formulation and proof.

Considering the exact sequence (5) in subsection 10.5, we see that $S\hat{X}$ has equivariant homotopy type of the mapping cone of Δ. If one would be interested in the stable homotopy type of $S\hat{X}$ then it is possible to use the S-equivalence $i_+ - i_-$: $W \to Y$. Then from (1), (2), (4) we see that $S\hat{X}$ has stable equivariant homotopy type of the mapping cone of the map

$$\nabla^1 \colon SW \wedge \mathbf{Z}^+ \to SW \wedge \mathbf{Z}^+$$

given (informally) by

$$\nabla^1 = 1 \wedge z \wedge 1 + t^{-1} \circ (1 \wedge \bar{z} \wedge 1). \tag{9}$$

Here z: $W \to W$ is the S-map from the n-isometry of W (see §2). To clarify (9), note that if z and $\bar{z}$ are realized by continuous maps then ∇^1 is given by the formula

$$\nabla^1(\tau \wedge w \wedge n) = \begin{cases} (2\tau) \wedge z(w) \wedge n & \text{for } 0 \leqslant \tau \leqslant 1/2, \\ (2\tau - 1) \wedge \bar{z}(w) \wedge (n-1) & \text{for } 1/2 \leqslant \tau \leqslant 1. \end{cases}$$

To define ∇^1 in general, the similar map $S^rW \wedge \mathbf{Z}^+ \to S^rW \wedge \mathbf{Z}^+$ for some large r should be constructed and then use the desuspension operator (in some stable category).

11. Main result (The formulation with Λ-quintets). In this section each simple even-dimensional knot will be assigned some Λ-quintet, i.e. a collection consisting of two Λ-modules, a homomorphism and two bilinear forms. The knot type defines the corresponding Λ-quintet up to isomorphism. Basing upon the main result of [**1**, Theorem 4.2] and upon Theorem 7.4 we obtain here the classification of simple even-dimensional spherical knots in terms of Λ-quintets. The results of §10 allow us to give a geometrical interpretation of modules from a Λ-quintet of a knot.

11.1. Let $K = (S^{2q+2}, k^{2q})$ be some $(q-1)$-simple spherical knot. Let $V^{2q+1} \subset S^{2q+2}$ be its arbitrary $(q-1)$-connected Seifert manifold. Consider $2q$-isometry (V, u, z) of V (defined in §2). Since V is $(q-1)$-connected and u is a duality map, then length $V \leqslant 2$. Applying the construction of subsection 4.3 we get the P-quintet $\mathfrak{q}$, corresponding to (V, u, z). Then let us assign the knot K the L-quintet $\tilde{\mathfrak{q}} = \mathfrak{q} \otimes_P L$ (see 7.2, 7.3).

The only arbitrariness in this construction is the choice of the Seifert manifold V. However, by Theorem 2.6, the R-equivalence class of (V, u, z) does not depend on the choice of V, but it is correctly determined by the knot type of K. Consequently, by virtue of Proposition 4.5, the R-equivalence class of $\mathfrak{q}$ is defined by the knot type K. By Theorem 7.4.I this implies that the knot type K defines the L-quintet $\tilde{\mathfrak{q}}$ up to isomorphism.

11.2. THEOREM. *If $q \geqslant 4$ then the map sending each knot to the corresponding L-quintet $\tilde{\mathfrak{q}}$ is a bijection of the set of $(q-1)$-simple $2q$-dimensional spherical knot types on the set of isomorphism classes of L-quintets of parity $(-1)^{q+1}$.*

PROOF. If any two $(q-1)$-simple $2q$-dimensional spherical knots K_1 and K_2 are assigned isomorphic L-quintets, then by Theorem 7.4.I the P-quintets corresponding to these knots are R-equivalent. By Proposition 4.5 the corresponding $2q$-isometries are R-equivalent too (for $q \geqslant 3$), and now by Theorem 2.6 the knots K_1 and K_2 are of the same isotopy type (for $q \geqslant 4$).

For any L-quintet $\tilde{\mathfrak{q}}$ of parity $(-1)^{q+1}$ there exists a P-quintet $\mathfrak{q}$ of the same parity such that $\tilde{\mathfrak{q}} = \mathfrak{q} \otimes_P L$ (by Theorem 7.4.II). By Proposition 4.4 the P-quintet $\mathfrak{q}$ can be realized by some $2q$-isometry $(\mathfrak{X}, u, z)$ with length $\mathfrak{X} \leqslant 2$. Since $q \geqslant 4$ we may use Theorem 2.6, and so the R-equivalence class of $(\mathfrak{X}, u, z)$ corresponds to some $(q-1)$-simple $2q$-dimensional spherical knot K. Clearly, the L-quintet $\tilde{\mathfrak{q}}$ is assigned to this knot K.

The theorem is proved.

We consider below the problem of geometrical interpretation of the invariants from the L-quintet of a knot. We show that they are homology invariants of the knot complement's infinite cyclic covering. Hence it will be better to pass from the ring L to its subring $\Lambda = \mathbf{Z}[t, t^{-1}]$ in which the generators have a clearer geometrical sense. The generators $z, z^{-1}, \bar{z}^{-1}$ of L are of auxiliary character and they should be excluded from the final result's formulation.

11.3. *A Λ-quintet* is defined as a collection (A, B, α, l, ψ) consisting of Λ-modules A and B, a Λ-homomorphism $\alpha: A \otimes \mathbf{Z}_2 \to B$, and forms $l: T(A) \otimes_{\mathbf{Z}} T(A) \to \mathbf{Q}/\mathbf{Z}$ and $\psi: B \otimes_{\mathbf{Z}} B \to \mathbf{Z}_4$, if the following conditions hold:

(a) A is of type K, i.e. it is finitely-generated over Λ and the multiplication by $t-1$ is an isomorphism $A \to A$;

(b) the sequence

$$0 \to A \otimes \mathbf{Z}_2 \xrightarrow{\alpha} B \xrightarrow{\beta} \operatorname{Hom}(A; \mathbf{Z}_2) \to 0$$

is exact, where $\beta(b)(a) = \psi(b \otimes \alpha(\pi(a)))$, $b \in B$, $a \in A$, and $\pi: A \to A \otimes \mathbf{Z}_2$ is the projection;

(c) the pairing l is nondegenerate;

(d) the pairings l and ψ are ε-symmetric;

(e) the composition

$$B \xrightarrow{\gamma} A \xrightarrow{\pi} A \otimes \mathbf{Z}_2 \xrightarrow{\alpha} B$$

coincides with the multiplication by 2. Here, for $b \in B$, $\gamma(b)$ is defined by $\psi(b \otimes \alpha(\pi(a))) = l(\gamma(b) \otimes a)$ to be satisfied for all $a \in T(A)$;

(f) $l(ta \otimes tb) = l(a \otimes b)$, $a, b \in T(A)$;

(g) $\psi(ta \otimes tb) = \psi(a \otimes b)$, $a, b \in B$.

The number ε, which can be $+1$ or -1, will be called *parity* of the Λ-quintet (A, B, α, l, ψ). Two Λ-quintets $(A_\nu, B_\nu, \alpha_\nu, l_\nu, \psi_\nu)$, $\nu = 1, 2$, are called *isomorphic* if there exist Λ-isomorphisms $f\colon A_1 \to A_2$, $\xi\colon B_1 \to B_2$ such that

$$l_1 = l_2 \circ \left(f|_{T(A_1)} \otimes f|_{T(A_1)} \right), \qquad \psi_1 = \psi_2 \circ (\xi \otimes \xi)$$

and the diagram

$$\begin{array}{ccc} A_1 \otimes \mathbf{Z}_2 & \xrightarrow{f\otimes 1} & A_2 \otimes \mathbf{Z}_2 \\ \downarrow \alpha_1 & & \downarrow \alpha_2 \\ B_1 & \xrightarrow[\xi]{} & B_2 \end{array}$$

is commutative.

11.4. Any L-quintet $\mathfrak{q} = (A, B, \alpha, l, \psi)$ is as well a Λ-quintet. In fact, Λ is a subring in L (we suppose here, as we earlier did, that $t = 1 - z^{-1}$) and so A and B are Λ-modules of type K. This shows that condition (a) of the P-quintet definition in subsection 11.3 is satisfied. Conditions (b)–(e) in the definitions of L-quintet and Λ-quintet coincide. To verify (f) we note that if $l(za \otimes b) = l(a \otimes \bar{z}b)$ for all $a, b \in T(A)$ then

$$\begin{aligned} l(ta \otimes tb) &= l\big((1 - z^{-1})a \otimes tb\big) = l(a \otimes tb) - l\big(z^{-1}a \otimes tb\big) \\ &= l(a \otimes tb) - l\big(z^{-1}a \otimes \bar{z}\bar{z}^{-1}tb\big) = l(a \otimes tb) - l\big(a \otimes \bar{z}^{-1}tb\big) \\ &= l\big(a \otimes (1 - \bar{z}^{-1})tb\big) = l(a \otimes b) \end{aligned}$$

since $t(1 - \bar{z}^{-1}) = 1$ in L. This proves (f). Similarly (g) follows.

Conversely, for any Λ-quintet (A, B, α, l, ψ) the modules A and B are of type K, i.e. the multiplication by $(1 - t)$ is an automrphism of each of these modules. Therefore these modules can be provided with L-module structures defining the multiplication by z as the map inverse to the multiplication by $(1 - t)$, i.e. $z = (1 - t)^{-1}$. It is clear that then α would be an L-homomorphism and conditions (a)–(e) from the L-quintet definition are satisfied. To verify (f) we have

$$\begin{aligned} l(za \otimes b) &= l\big(za \otimes (1 - t)zb\big) = l(za \otimes zb) - l(za \otimes tzb) \\ &= l(tza \otimes tzb) - l(za \otimes tzb) = l\big((t - 1)za \otimes tzb\big) \\ &= -l(a \otimes tzb) = l(a \otimes \bar{z}b), \end{aligned}$$

since $\bar{z} = -tz$ in L. Here $a, b \in T(A)$. Similarly (g) follows.

Thus any L-quintet is a Λ-quintet and any Λ-quintet defines unique L-quintet. Clearly isomorphic L-quintets define isomorphic Λ-quintets and conversely.

11.5. Comparing the results of subsections 11.1 and 11.4, we see that any $(q - 1)$-simple $2q$-dimensional spherical knot $K = (S^{2q+2}, k^{2q})$ is assigned some Λ-quintet (A, B, α, l, ψ) of parity $(-1)^{q+1}$, which is defined by the knot type up to isomorphism. According to the construction the module A is isomorphic (over L, and so over Λ as well) to the module $H_q(V) \otimes_P L$ for any Seifert manifold V of K. Applying Theorem 10.2(2) we obtain the Λ-isomorphism

$$A \approx H_q(\hat{X}), \tag{1}$$

where $\hat{X}$ is the reduced space of the infinite cyclic covering. Similarly there exists a Λ-isomorphism

$$B \approx \sigma_{q+2}(\hat{X}). \tag{2}$$

Let us notice that the Λ-module structure in $H_q(\hat{X})$ and $\sigma_{q+2}(\hat{X})$ is given by $ta = t_*(a)$, where $t\colon \hat{X} \to \hat{X}$ is the generator of the covering transformation group chosen in subsection 10.1.

If we use (1) and (2) then α is a homomorphism

$$\alpha\colon H_q(\hat{X}) \otimes_{\mathbf{Z}} \mathbf{Z}_2 \to \sigma_{q+2}(\hat{X})$$

which acts as follows: it is the composition of the homomorphism

$$h_q^{-1} \otimes 1\colon H_q(\hat{X}) \otimes \mathbf{Z}_2 \to \sigma_q(\hat{X}) \otimes \mathbf{Z}_2$$

where h_q is the Hurewicz homomorphism (which is an isomorphism here since $\hat{X}$ is $(q-1)$-connected), followed by the homomorphism

$$\sigma_q(\hat{X}) \otimes \mathbf{Z}_2 \to \sigma_{q+2}(\hat{X})$$

sending $a \otimes 1$ to $a \circ \eta^2$ where $a \in \sigma_q(\hat{X})$ and $\eta^2 \in \sigma_{q+2}(S^q)$ is the single nonzero element.

Thus, Theorem 11.2 and the notes of subsections 11.4 and 11.5 imply the following statement which is the summarizing result of the paper.

11.6. THEOREM. I. *For $q \geqslant 4$ any $(q-1)$-simple $2q$-dimensional spherical knot has the collection of invariants consisting of two Λ-modules $A = H_q(\hat{X})$ and $B = \sigma_{q+2}(\hat{X})$, a Λ-homomorphism $\alpha\colon A \otimes \mathbf{Z}_2 \to B$ (which was geometrically interpreted in* 11.5), *and two bilinear forms $l\colon T(A) \otimes T(A) \to \mathbf{Q}/\mathbf{Z}$, $\psi\colon B \otimes B \to \mathbf{Z}_4$. Here $\hat{X}$ is the reduced space of the knot's infinite cyclic covering (see* 10.1), *$\sigma_{q+2}(\hat{X})$ denotes the $(q+2)$-dimensional stable homotopy group, and $T(A) = \operatorname{Tors}_{\mathbf{Z}} A$.*

II. *for any such knot the collection (A, B, α, l, ψ) is a Λ-quintet of parity $(-1)^{q+1}$ (i.e. it satisfies conditions* (a)–(g) *of subsection* 11.3). *The knot type defines this Λ-quintet up to isomorphism.*

III. *For $q \geqslant 4$ any two $(q-1)$-simple $2q$-dimensional spherical knots are equivalent if their Λ-quintets are isomorphic.*

IV. *For $q \geqslant 4$ any Λ-quintet of parity $(-1)^{q+1}$ is Λ-quintet of some $(q-1)$-simple $2q$-dimensional spherical knot.*

Let us notice that the invariants of statement I of this theorem are also defined for $q < 4$, but in this case they do not constitute a complete system of invariants. Statement II is true for $q < 4$ as well.

The construction of the form l not using the choice of Seifert manifold was proposed earlier in the works of J. Levine [**11**, **12**] and independently in the author's works [**3**, **4**]. The equivalence of the definition of l given here and the definition of [**3**, **4**] follows from results of §7 and [**4**].

11.7. In some cases it is possible to simplify the invariants system (A, B, α, l, ψ) constituting the Λ-quintet of a simple even-dimensional spherical knot. For example, if A is finite and has no 2-torsion then it is clear that $B = \alpha = \psi = 0$ and so a complete system of invariants is given by the Λ-module A and the form l: $A \otimes A \to \mathbf{Q}/\mathbf{Z}$. This case was studied in **[9]**.

Suppose now that A is finite and has exponent 2. From conditions (b), (c), (e) of subsection 11.3 it follows that A is isomorphic to $2B$ and so B is isomorphic to the direct sum of groups $\mathbf{Z}_4$ with the number of summands equal to the dimension of A over $\mathbf{Z}_2$. If $b, b' \in B$ then

$$l(\gamma(b) \otimes \gamma(b')) = \psi(b \otimes \alpha(\pi(\gamma(b')))) = \psi(b \otimes 2b') = 2\psi(b \otimes b').$$

Since γ: $B \to A$ is, in this case, an epimorphism, then this equality shows that ψ defines l. Thus in this case the complete system of invariants consists of Λ-module B and the pairing ψ: $B \otimes B \to \mathbf{Z}_4$ with the following properties: (1) $2B = \{b \in B;\ 2b = 0\}$; (2) ψ is nondegenerate (by Lemma 3.7); (3) ψ is $(-1)^{q+1}$-symmetric; (4) $\psi(ta \otimes tb) = \psi(a \otimes b)$ for all $a, b \in B$. Conversely, if B and ψ satisfying these conditions are given, then we can put $A = 2B$, $\alpha =$ inclusion and define l as it was pointed out above. As a result we obtain some Λ-quintet.

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